Familles évanescentes de compacts et asymptotique de la formule des traces Colloque en l'honneur du 60$^{\text{e}}$anniversaire de Jean-Pierre Labesse

François Sauvageot

26 septembre 2003

1 Codes de Goppa et borne de Drinfled-Vlăduţ

1.1 On considère un triplet (X, D, G) formé d'une courbe projective lisse X sur $\mathbf{F}_q$, D un diviseur de la forme $D = \sum_{i=1}^n P_i$ où les P_i sont des points de $X(\mathbf{F}_q)$ tous distincts et G est un diviseur $\mathbf{F}_q$-rationnel de support disjoint de celui de D. On note g le genre de X et a le degré de G.

On suppose $2g - 2 < a \leq n + g - 1$ et on pose $C = H^0(X, \Omega_x^1(D - G))$. On a une application injective (du fait que $H^0(X, \Omega_x^1(-G)) = 0$)

$$res_D : C \to \mathbf{F}_q^n$$

qui à tout ω associe ses résidus en $(P_1, \ldots, P_n)$. On a défini ainsi un code, i.e. un sous-espace vectoriel de $\mathbf{F}_q^n$. Le poids de w (le nombre de ses coordonnées non nulles) est l'ordre du diviseur de ses pôles. C'est donc au moins $d = a - (2g - 2)$. La dimension k de C est donnée par le théorème de Riemann-Roch : $k = n + g - 1 - a$. En particulier $k + d = n(1 - g/n + 1/n)$. Asymptotiquement le taux de transmission (k/n) et le poids relatif (d/n) sont donc gouvernés par la quantité n/g. Plus n/g est grand, plus le code ainsi construit est intéressant.
1.2 On note $A(q) = \limsup n/g$. Drinfeld et Vlăduţ ont montré, en rafinant les travaux de Ihara, que $A(q) \leq \sqrt{q} - 1$ (l'hypothèse de Riemann ne fournit que la majoration $A(q) \leq 2\sqrt{q}$). En effet, notons N(X,r) le nombre de points de X sur $\mathbf{F}_{q^r}$, $\sqrt{q}\alpha_i$ les valeurs propres de l'opérateur de Frobenius agissant sur la jacobienne de X. On a, pour t et u dans $\mathbf{N}$,

$$\begin{aligned} 0 &\leq \sum_{i=1}^{2g} \left| \sum_{j=0}^{t} \alpha_i^j \right| \\ &\leq \sum_{i=1}^{2g} \left(t + 1 + \sum_{j=1}^{t} (t + 1 - j)(\alpha_i^j + \bar{\alpha}_i^j \right) \\ &\leq 2g(t + 1) + 2 \sum_{j=1}^{t} (t + 1 - j) q^{-j/2} \left(\mathrm{N}(\mathrm{X}, j) - q^j - q^{-j} \right) \end{aligned}$$

et donc

$$\sum_{j=1}^{t}\left(1-\frac{j}{t+1}\right)\frac{\mathrm{N}(\mathrm{X},j)}{q^{j/2}g}\leq 1+\sum_{j=1}^{t}\left(1-\frac{j}{t+1}\right)\frac{q^{j/2}+q^{-j/2}}{g}\leq 1+\sum_{j=1}^{t}\frac{q^{j/2}+q^{-j/2}}{g}\ .$$

En particulier, pour $t=o(\log(g))$, le membre de droite est équivalent à 1. Notons $\mathrm{B}(\mathrm{X},m)$ le nombre des points de degré m de sorte que $\mathrm{N}(\mathrm{X},r)=\sum_{m|r}m\mathrm{B}(\mathrm{X},m)$. On a

$$\sum_{j=1}^{u}\frac{\mathrm{N}(\mathrm{X},j)}{q^{j/2}g}=\sum_{m=1}^{u}\frac{m\mathrm{B}(\mathrm{X},m)}{q^{m/2}-1}\frac{1-q^{-m[u/m]/2}}{g}$$

et en particulier, si $1=o(u)$,

$$\frac{\mathrm{N}(\mathrm{X},1)}{g(\sqrt{q}-1)}\simeq\frac{\mathrm{N}(\mathrm{X},1)(1-q^{-u/2})}{g(\sqrt{q}-1)}\leq\sum_{j=1}^{u}\frac{\mathrm{N}(\mathrm{X},j)}{q^{j/2}g}\ .$$

Or, si $u=o(t)$,

$$\sum_{j=1}^{u}\frac{\mathrm{N}(\mathrm{X},j)}{q^{j/2}g}\simeq\sum_{j=1}^{u}\left(1-\frac{u}{t+1}\right)\frac{\mathrm{N}(\mathrm{X},j)}{q^{j/2}g}\leq\sum_{j=1}^{t}\left(1-\frac{j}{t+1}\right)\frac{\mathrm{N}(\mathrm{X},j)}{q^{j/2}g}\simeq 1$$

et il vient

$$\frac{\mathrm{N}(\mathrm{X},1)}{g(\sqrt{q}-1)}\leq 1\ .$$

Cette borne est optimale lorsque q est un carré, comme l'ont montré Ihara ainsi que Tsfasman-Vlăduţ-Zink en comptant les courbes supersingulières dans $\mathrm{X}_0(\ell)$ pour ℓ tendant vers l'infini. Pour q non carré, cette question est encore ouverte (en particulier pour $q=2$).

2 Extension des résultats de Serre

2.1 Serre a retrouvé ce résultat en prouvant *a priori* que les valeurs propres de l'opérateur de Hecke normalisé $T'_p=p^{1/2}T_p$ opérant dans l'espace des formes modulaires paraboliques de poids $2k$ pour $\Gamma_0(N)$ sont asymptotiquement (lorsque $N+2k$ tend vers l'infini) équiréparties selon la restriction de la mesure de Plancherel sur le dual de $\mathrm{PGL}_2(\mathbf{Q}_p)$ à l'ensemble des séries principales non ramifiées. Le résultat s'en déduit alors pour $q=p^2$ en faisant tendre N vers l'infini.
2.2 On peut généraliser cette approche en travaillant dans un cadre général et montrer que les opérateurs de Hecke contribuent asymptotiquement à la partie géométrique de la formule des traces seulement par des intégrales orbitales sur des éléments centraux. Cette remarque est appliquée pour calculer la caractéristique d'Euler-Poincaré de variétés de Shimura, grâce aux travaux d'Arthur, et leur nombre de points, en utilisant une formule conjecturale due à Kottwitz et qui est un théorème pour les courbes.
2.3 Soit E un corps totalement réel inclus dans $\mathbf{C}$, de degré m sur $\mathbf{Q}$, p un nombre premier inerte dans E et L une algèbre de quaternion sur E, déployé en

$v = p$ et en une unique place réelle. Soit G le groupe $\mathrm{G} = \mathrm{Res}_{\mathrm{E}/\mathbf{Q}}\mathrm{GL}_1(\mathrm{L})$; on considère

$$\mathrm{X}(\mathrm{G},\mathrm{K}_\infty) = \mathrm{G}(\mathrm{F})\backslash\mathrm{G}(\mathbf{A})/\mathrm{K}_\infty$$

où K_∞ est un sous-groupe d'indice fini de $\underline{\mathrm{K}}_\infty\mathrm{A}^0_{\mathrm{G},\infty}$, où $\mathrm{A}^0_{\mathrm{G},\infty}$ est la composante connexe de l'identité dans $\mathrm{A}_\mathrm{G}(\mathbf{R})$ et $\underline{\mathrm{K}}_\infty$ un compact maximal fixé de G_∞. Ici $\underline{\mathrm{K}}_\infty\mathrm{A}^0_{\mathrm{G},\infty} \simeq \mathrm{GO}(2,\mathbf{R}) \times \mathrm{GL}(1,\mathbf{H})^{m-1}$.

Si K est un sous-groupe compact ouvert de G_f, on écrit $\mathrm{X}(\mathrm{G},\mathrm{K}_\infty\mathrm{K})$ pour $\mathrm{X}(\mathrm{G},\mathrm{K}_\infty)/\mathrm{K}$.

Soit Sh(G) la variété de Shimura définie par une paire (G, X) où X est une classe de G_∞-conjugaison de morphismes de $\mathrm{Res}_{\mathbf{C}/\mathbf{R}}\mathrm{G}_m$ dans G. Soit K_∞ le centralisateur de X, de sorte que $\mathrm{Sh}(\mathrm{G})(\mathbf{C}) = \mathrm{X}(\mathrm{G},\mathrm{K}_\infty)$. Il y a une structure naturelle Sh(G,K) de varieté quasi-projective non connexe sur $\mathbf{C}$ sur $\mathrm{X}(\mathrm{G},\mathrm{K}_\infty\mathrm{K})$. Dans le cadre étudié, on a $\mathrm{K}_\infty \simeq \mathrm{GO}^+(2,\mathbf{R})\times\mathrm{GL}(1,\mathbf{H})^{m-1}$ et donc $[\underline{\mathrm{K}}_\infty\mathrm{A}^0_{\mathrm{G},\infty} : \mathrm{K}_\infty] = 2$.

On note $\chi(\mathrm{G},\mathrm{K})$ la caractéristique d'Euler-Poincaré pour la cohomologie L^2 de Sh(G, K), g_K son genre, $\mathrm{N}(\mathrm{G},\mathrm{K},\mathbf{F}_q)$ et $\overline{\mathrm{N}}(\mathrm{G},\mathrm{K},\mathbf{F}_q)$ les nombres de points sur $\mathbf{F}_q$ avec $q = p^{mr}$ de Sh(G, K) et de sa complétion respectivement.

2.4 H. Reimann a démontré [Rei97, Proposition 10.9] que le nombre de points de Sh(G,K) sur $\mathbf{F}_q$, pour $q = p^{mr}$ est donné par la trace d'une fonction f dans $C^\infty_c(\mathrm{L}(\mathbf{A})/\mathrm{E}_\infty)$ opérant sur l'espace des formes automorphes sur $\mathrm{L}(\mathbf{Q})\mathrm{E}_\infty\backslash\mathrm{L}(\mathbf{A})$. Notamment

$$f_v = \psi_{r,0} - (p^m - 1)\sum_{0<k\leq r/2}\psi_{r-k,k}\ ,$$

où $\psi_{j,k}$ est la fonction caractéristique de la double classe de $\begin{pmatrix} p^j & 0 \\ 0 & p^k \end{pmatrix}$ modulo $\mathrm{GL}_2(\mathcal{O}_v)$ dans $\mathrm{GL}_2(\mathrm{E}_v)$.

Heuristiquement une famille évanescente de sous-groupes compacts de G^p_f est une famille pour laquelle les seuls termes intervenant asymptotiquement dans les formules de traces calculant la caractéristique d'Euler-Poincaré et le nombre de points sur $\mathbf{F}_q$ sont ceux correspondant à des éléments rationnels. Plus précisément : pour tout compact C de G_p, il existe un sous-ensemble fini $\mathrm{Z}_{\mathcal{K},\mathrm{C}}$ de $\mathrm{Z}_\mathrm{G}(\mathrm{F})$, tel que, pour tout x dans $\mathrm{Z}_\mathrm{G}(\mathrm{F})$, elliptique dans G_∞,

$$\lim_{\mathrm{K}\in\mathcal{K}} 1_{\mathrm{K}\times\mathrm{C}}(x) = 1_{\mathrm{Z}_{\mathcal{K},\mathrm{C}}}(x)\ .$$

Théorème 1 *Si r est pair et $\mathrm{Z}_{\mathcal{K},p^{r/2}\underline{\mathrm{K}}_p}$ n'est pas vide, alors*

$$\lim_{\mathrm{K}\in\mathcal{K}} \frac{\mathrm{N}(\mathrm{G},\mathrm{K},\mathbf{F}_q)}{\chi(\mathrm{G},\mathrm{K})} = \frac{1-p^m}{2}\ .$$

Sinon la limite est nulle.

Démonstration : On a

$$\lim_{\mathrm{K}\in\mathcal{K}} \frac{\mathrm{N}(\mathrm{G},\mathrm{K},\mathbf{F}_q)}{\chi(\mathrm{G},\mathrm{K})} = \lim_{\mathrm{K}\in\mathcal{K}} \frac{\mathrm{vol}(\mathrm{K})\mathrm{N}(\mathrm{G},\mathrm{K},\mathbf{F}_q)}{\mathrm{vol}(\mathrm{K})\chi(\mathrm{G},\mathrm{K})}$$

et chacun des termes ont des limites données par des formules de traces pour des fonctions différant aux places p et ∞.

Le numérateur est un multiple de $-(p^m-1)\,\mathrm{Card}(\mathrm{Z}_{\mathcal{K},p^{r/2}\underline{\mathrm{K}}_p})$ ou 0, tandis que le dénominateur est le même multiple de $2\chi(\mathrm{G})\,\mathrm{Card}(\mathrm{Z}_{\mathcal{K},\underline{\mathrm{K}}_p})$ (avec $2 = [\underline{\mathrm{K}}_\infty \mathrm{A}^0_{\mathrm{G},\infty} : \mathrm{K}_\infty]$).

Comme $\mathrm{Z}_{\mathcal{K},\underline{\mathrm{K}}_p}$ contient au moins 1 et que $\mathrm{Z}_{\mathcal{K},p^{r/2}\underline{\mathrm{K}}_p}$ est soit vide, soit un translaté de $\mathrm{Z}_{\mathcal{K},\underline{\mathrm{K}}_p}$, le résultat en découle. □

Corollaire 1 *Si* $r=2$, $\mathrm{Z}_{\mathcal{K},p\underline{\mathrm{K}}_p}$ *n'est pas vide et, pour tout* K *dans* $\mathcal{K}$*,* $\mathrm{Sh}(\mathrm{G},\mathrm{K})$ *est connexe, alors*

$$\lim_{\mathrm{K}\in\mathcal{K}} \frac{\overline{\mathrm{N}}(\mathrm{G},\mathrm{K},\mathbf{F}_q)}{g_{\mathrm{K}}} = \sqrt{q}-1\ .$$

Exemple – Les conditions sont vérifiées dans le cas traité par Serre.

3 Asymptotique des variétés de Shimura

3.1 On peut énoncer les résultats précédents dans un cadre plus général. Soit G un groupe algébrique, connexe, réductif sur $\mathrm{F}=\mathbf{Q}$. On suppose que G contient un tore maximal défini sur $\mathbf{R}$, anisotrope modulo A_{G} (le tore déployé maximal de Z_{G}), que G_{der} est simplement connexe et que le tore $\mathbf{R}$-déployé maximal de Z_{G} est déployé sur $\mathbf{Q}$.

Fix $\underline{\mathrm{K}} = \prod_v \underline{\mathrm{K}}_v$ a maximal compact subgroup of $\mathrm{G}(\mathbf{A})$, hyperspecial in any finite place where G is unramified, and μ_v the Haar mesure on $\mathrm{G}_v = \mathrm{G}(\mathrm{F}_v)$ which gives mass 1 to $\underline{\mathrm{K}}_v$. Let $\mathcal{H}_\infty$ be the set of smooth, compactly supported, $\underline{\mathrm{K}}_\infty$-finite functions on $\mathrm{G}_\infty = \mathrm{G}(\mathrm{F}\otimes\mathbf{R})$ and $\mathcal{H}_f$ the set of smooth, compactly supported functions on $\mathrm{G}_f = \mathrm{G}(\mathbf{A}_f)$.

We write G_γ for the centralizer of γ in G. For h in $\mathcal{H}_f$, define the orbital integral

$$\mathcal{O}^{\mathrm{G}_f}_\gamma(h) = \int_{\mathrm{G}_\gamma(\mathbf{A}_f)\backslash \mathrm{G}(\mathbf{A}_f)} h(x^{-1}\gamma x)dx$$

for an arbitrary fixed measure on the orbit. And for MN a rational Levi decomposition of a rational parabolic subgroup of G, let h_{M} be defined on M_f by

$$h_{\mathrm{M}}(x) = \int_{\underline{\mathrm{K}}_f}\int_{\mathrm{N}_f}\int_{\mathrm{M}_{x,f}\backslash \mathrm{M}_f} h(k^{-1}m^{-1}xmnk)dm.dn.dk\ .$$

3.2 Let K be compact open subgroup of G_f and write h for its normalized characteristic function. By [Art89, Theorem 6.1] (but see also [GKM97, 7.19]), the Lefschetz number $\chi_{\mathrm{V}}(\mathrm{X}(\mathrm{G},\mathrm{K}))$ is a finite linear combination of terms of the form $h_{\mathrm{M}}(\gamma)$ for M and γ in finite sets (depending only on the support of h).

$$\sum_{M\in\mathcal{L}(G)} (-1)^{\dim(A_M/A_G)} \frac{|W_0^M|}{|W_0^G|} \sum_{\gamma\in(M(\mathbf{Q}))} \chi(M_\gamma)|\iota^M(\gamma)|^{-1}\Phi_M(\gamma,V)f_M(\gamma)\ ,$$

où la somme en γ est une somme sur l'ensemble des $M(\mathbf{Q})$-classes de conjugaison d'éléments semi-simples dans $M(\mathbf{Q})$ (d'après [Art89 Theorem 5.1]).

3.3 Now fix a prime p and a place v of E unramified over p. We suppose that G is unramified over p, that the weight of $\mathrm{Sh}(\mathrm{G})$ is defined over $\mathbf{Q}$ and that $\mathrm{Sh}(\mathrm{G})/\underline{\mathrm{K}}_p$ has a canonical integral model relative to v in the sense of [Mil92, Definition 2.9]. If $\mathbf{F}_q$ is a finite field containing the residual field of E_v and K is an open compact subgroup of $G^{p,\infty}$, $\mathrm{Sh}(G,\underline{K}_p K)$ can be regarded as being defined over $\mathbf{F}_q$ and we can compute $N(G,K,q)$, its number of points of over $\mathbf{F}_q$, thanks to [Kot88, formula 3.1], at least when K is small enough and when [Mil92, Main conjecture 4.4] is true :

$$N(G,K,q)=\sum_{(\gamma_0;\gamma,\delta)} c(\gamma_0)\frac{\mathrm{vol}\,(I_{\gamma_0,\gamma,\delta}(\mathbf{Q})\backslash I_{\gamma_0,\gamma,\delta}(\mathbf{A}))}{\mathrm{vol}(K)}\mathcal{O}_\gamma^{G^{p,\infty}}(1_K)TO_\delta(\phi_q)\ ,$$

the sum being over effective Frobenius triplets $(\gamma_0;\gamma,\delta)$, where $I_{\gamma_0,\gamma,\delta}$ is a certain inner form of G_{γ_0}, ϕ_q is the characteristic function of a certain double class Y_p in the group of points of G over the fraction field of the ring of Witt vectors over $\mathbf{F}_q$ and where TO_δ stands for the twisted orbital integral over the orbit of δ. (We refer to [Mil92] for details.)

4 Familles évanescentes de compacts

4.1 For a countable family $\mathcal{K}$, we define limits for K in $\mathcal{K}$ as taken along the filter of complements of finite sets.

Définition 1 *A countable family $\mathcal{K}$ of compact sets in G_f is called vanishing if*

1. *The set $\cup_{K\in\mathcal{K}}K$ is relatively compact.*
2. *There exists a finite subset $Z_{\mathcal{K}}$ of $Z_G(F)$, such that, for any x in $Z_G(F)$, elliptic in G_∞,*
$$\lim_{K\in\mathcal{K}} 1_K(x)=1_{Z_{\mathcal{K}}}(x)\ .$$
3. *For all rational non-central element γ in G,*
$$\lim_{K\in\mathcal{K}}\mathcal{O}_\gamma^{G_f}(1_K)=0\ .$$
4. *For all (M,γ) with M a proper Levi subgroup of G and γ a rational element in M*
$$\lim_{K\in\mathcal{K}}(1_K)_M(\gamma)=0\ .$$

We will say that the family is locally vanishing if we can substitute to 3. the stronger condition :

3'. For all rational non-central element γ_0 in G and all $\gamma=(\gamma_v)_v$ in G_f such that, for any finite place v of F, γ_v is stably conjugate to γ_0 in $G(F_v)$,

$$\lim_{K\in\mathcal{K}}\mathcal{O}_\gamma^{G_f}(1_K)=0\ .$$

4.2 Let S be a finite set of places of F. By taking product over places in S or restricted product over places outside S, we get F_S, $G_S=G(F_S)$ and $\mathbf{A}^S$, $G^S=G(\mathbf{A}^S)$. Let S be a finite set of finite places of F we will say that a countable family $\mathcal{K}$ of compact sets in G_f^S (resp. G_S) is vanishing if, for any compact set C in G_S (resp. in G_f^S), the family $(K\times C)_{K\in\mathcal{K}}$ is vanishing and condition *3.* holds with G_f replaced by G_f^S (resp. G_S). We write $Z_{\mathcal{K},C}$ for the finite subset of $Z_G(F)$ refered to by condition *2.*. At last, we similarly define the notion of a family locally vanishing by replacing in *3'.* G_f by G_f^S (resp. G_S) and taking v in finite places of F outside of S (resp. in S).

Theorem 4.3 *Let $\mathcal{K}$ be a vanishing family of compacts sets in G_f such that ξ_{V} is trivial on $\mathrm{Z}_{\mathcal{K}}$, then*

$$\lim_{\mathrm{K}\in\mathcal{K}} \mathrm{vol}(\mathrm{K})\chi_{\mathrm{V}}(\mathrm{X}(\mathrm{G},\mathrm{K})) = [\underline{\mathrm{K}}_\infty \mathrm{A}^0_{\mathrm{G},\infty} : \mathrm{K}_\infty]\chi(\mathrm{G})\deg(\mathrm{V})\,\mathrm{Card}(\mathrm{Z}_{\mathcal{K}}) ,$$

where $\chi(\mathrm{G})$ is the constant introduced in [Art89].

Démonstration : To begin with, suppose $\mathrm{K}_\infty = \underline{\mathrm{K}}_\infty \mathrm{A}^0_{\mathrm{G},\infty}$ and let K' be acompact subset of G_f containing all the elements of $\mathcal{K}$. By definition of a vanishing family and thanks to Lebesgue dominated convergence theorem, only the terms with $\mathrm{M} = \mathrm{G}$ and γ in $\mathrm{Z}_{\mathcal{K}}$ contribute to the limit. We get, using Arthur's notations ([Art89])

$$\lim_{\mathrm{K}\in\mathcal{K}} \mathrm{vol}(\mathrm{K})\chi_{\mathrm{V}}(\mathrm{X}(\mathrm{G},\mathrm{K})) = \chi(\mathrm{G}) \sum_{z\in \mathrm{Z}_{\mathcal{K}}} |\iota^{\mathrm{G}}(z)|^{-1}\Phi_{\mathrm{G}}(z,\mathrm{V})h_{\mathrm{G}}(z) ,$$

where, for z in $\mathrm{Z}_{\mathcal{K}}$, we have $h_{\mathrm{G}}(z) = 1$, $\Phi_{\mathrm{G}}(z,\mathrm{V}) = \deg(\mathrm{V})\xi_{\mathrm{V}}(z) = \deg(\mathrm{V})$ and $|\iota^{\mathrm{G}}(z)| = 1$ by connectedness of G, hence the result in the case considered. The first remark following [Art89, Theorem 6.1] asserts that passing to subgroups of finite index multiplies the Lefschetz number by this index, hence the theorem. □

Write C_p for the compact set image of Y_p by the norm map.

Theorem 4.4 *Suppose Milne main conjecture is true for* G *and let $\mathcal{K}$ be a vanishing family of compacts sets in G^p_f, then*

$$\lim_{\mathrm{K}\in\mathcal{K}} \mathrm{vol}(\mathrm{K})\mathrm{N}(\mathrm{G},\mathrm{K},\mathbf{F}_q) = -\chi(\mathrm{G}) \sum_{(\gamma_0;\gamma,\delta)} TO_\delta(\phi_q) ,$$

the sum being over effective Frobenius triplets $(\gamma_0;\gamma,\delta)$ with γ_0 in $\mathrm{Z}_{\mathcal{K},\mathrm{C}_p}$.

Démonstration : The formula given above for $\mathrm{N}(\mathrm{G},\mathrm{K},\mathbf{F}_q)$ is a finite sum. We deduce from [Kot86, Proposition 8.2] and from the finitess of $H^1(\mathbf{Q}_p,\mathrm{G})$ that the summation index can be taken independently of K in $\mathcal{K}$. Note that, by definition of a Frobenius triplet, γ_0 is elliptic in G_∞. Hence by definition of a vanishing family and by Lebesgue dominated convergence theorem, only effective triplets with γ_0 in $\mathrm{Z}_{\mathcal{K},\mathrm{C}_p}$ contribute to the limit. For such elements, $c(\gamma_0) = 1$ and $\mathrm{vol}\,(\mathrm{I}_{\gamma_0,\gamma,\delta}(\mathbf{Q})\backslash \mathrm{I}_{\gamma_0,\gamma,\delta}(\mathbf{A})) = -\chi(\mathrm{G})$ by the second remark following [Art89, Theorem 6.1]. Whence the result. □

5 Exemples de familles évanescentes de compacts

Theorem 5.1 *Let* F *be a global field of characteristic zero,* S *a finite set of finite places of* F *and* A *a closed subset of an essential parabolic subgroup of* G_{S}*. If $\mathcal{K}$ is a countable family of compact subgroups of $\underline{\mathrm{K}}_{\mathrm{S}}$ such that*

$$\forall x \in \mathrm{G}_{\mathrm{S}} \qquad \lim_{\mathrm{K}\in\mathcal{K}} 1_{\mathrm{K}}(x) = 1_{\mathrm{A}}(x)$$

then $\mathcal{K}$ is a locally vanishing family of compact subgroups in G_{S}.

Démonstration : Les us check the properties required by definition 1. By assumption on $\mathcal{K}$, any K in $\mathcal{K}$ is contained in $\underline{\mathrm{K}}_{\mathrm{S}}$, hence the first property. This also ensures that A is a compact subset of $\underline{\mathrm{K}}_{\mathrm{S}}$. Hence, if T_∞ denotes a maximal elliptic torus in G_∞, for any compact subset $\mathrm{C}_f^{\mathrm{S}}$ of $\mathrm{G}_f^{\mathrm{S}}$, $\mathrm{Z}_{\mathcal{K},\mathrm{C}} = (\mathrm{A} \times \mathrm{C}_f^{\mathrm{S}} \times \mathrm{T}_\infty) \cap \mathrm{Z}_{\mathrm{G}}(\mathrm{F})$ is finite.

Now, if γ_0 is rational but not central and v is a place in S, any stable conjugate of γ_0 in G_v is not central either. So, by Lebesgue dominated convergence theorem and corollary 6.5, property 3 is satisfied.

Finally property 4 is a consequence of Lebesgue dominated convergence theorem and corollary 6.6. □

Example. – When G is GL_2 over $\mathbf{Q}$, examples of families are given by classical congruence subgroups $\Gamma(p^n)$, $\Gamma_0(p^n)$ or $\Gamma_1(p^n)$ for a fixed prime p and n going to infinity.

Theorem 5.2 *Let* F *be a global field of characteristic 0,* S *a finite set of finite places of* F *and* $\mathcal{K}$ *a countable family of open compact subgroups of* $\underline{\mathrm{K}}_f^{\mathrm{S}}$ *of the form* $\mathrm{K} = \Pi_v \mathrm{K}_v$ *where* K_v *is a parahoric subgroup of* G_v *(almost everywhere equal to* $\underline{\mathrm{K}}_v$*). Suppose that there is a family* $(v_{\mathrm{K}})_{\mathrm{K}\in\mathcal{K}}$ *of places of* F *such that* $v_{\mathrm{K}} \in \mathrm{S}$*, the reduction modulo* $\mathfrak{p}_{v_{\mathrm{K}}}$ *of* $\mathrm{K}_{v_{\mathrm{K}}}$ *is an essential parabolic and*

$$\lim_{\mathrm{K}\in\mathcal{K}} |\mathfrak{o}_{v_{\mathrm{K}}}/\mathfrak{p}_{v_{\mathrm{K}}}| = \infty$$

then $\mathcal{K}$ *is a locally vanishing family of compacts in* $\mathrm{G}_f^{\mathrm{S}}$*.*

Démonstration : Les us check the properties required by definition 1. By assumption on $\mathcal{K}$, any K in $\mathcal{K}$ is contained in $\underline{\mathrm{K}}_f^{\mathrm{S}}$, hence the first property. Now, if T_∞ is a maximal elliptic torus in G_∞ and C_{S} a compact set in G_{S}, since elements of $\mathcal{K}$ are parahoric subgroups, $\mathrm{Z}_{\mathcal{K},\mathrm{C}} = \mathrm{Z}_{\mathrm{G}}(\mathrm{F}) \cap (\underline{\mathrm{K}}_f^{\mathrm{S}} \times \mathrm{C}_{\mathrm{S}} \times \mathrm{T}_\infty)$ is finite.

Now let γ_0 be a non central element in G(F). Since $Ad(\gamma_0)$ is not the identity map, it can be the identity modulo $\mathfrak{p}_v$ only for a finite set of finite places v of F. Hence γ_0 is a non central element modulo $\mathfrak{p}_v$ for almost all finite places v of F.

So, for almost all K in $\mathcal{K}$, $\underline{\mathrm{K}}_{v_{\mathrm{K}}}$ is hyperspecial, γ_0 is not central modulo $\mathfrak{p}_{v_{\mathrm{K}}}$ and any stable conjugate of γ_0 in $\mathrm{G}_{v_{\mathrm{K}}}$ is not central modulo $\mathfrak{p}_{v_{\mathrm{K}}}$ either.

Hence the two last properties result from corollaries 7.4 and 7.5. □

Example. – When G is GL_2 over $\mathbf{Q}$, examples of families are given by classical congruence subgroups $\Gamma(n)$, $\Gamma_0(n)$ or $\Gamma_1(n)$ for n going to infinity.

6 Démonstrations – Paraboliques essentiels

6.1 In this section and the next F is a local field of characteristic 0, except when noted, G is connected reductive algebraic group over F and groups over F are equipped with arbitrary fixed Haar measures. If T is a maximal F-torus in G, let $\Phi(\mathrm{T},\mathrm{G})$ be the set of roots of G with respect to T and, for α in $\Phi(\mathrm{T},\mathrm{G})$, let $\mathrm{U}_{\mathrm{T},\alpha}$ be the unique T-stable subgroup of G with Lie algebra $\mathfrak{g}_\alpha$.

Définition 2 *A rational parabolic subgroup* P *of* G *is called essential if there exists a maximal* F*-torus* T *contained in a rational Levi subgroup* M *of* P *such that* $\Phi(\mathrm{T},\mathrm{M})$ *does not contain any connected component of* $\Phi(\mathrm{T},\mathrm{G})$*.*

Remark. – A minimal rational parabolic subgroup is essential. Moreover if G_{der} is simple, then any proper rational parabolic subgroup is essential.

Lemma 6.2 *Let* P *be an essential parabolic subgroup of* G*. The intersection of all parabolics conjugated to* P *is equal to* Z_G*.*

Démonstration : This intersection is a normal subgroup whose projection on any simple quotient of G is trivial. Hence it is contained in Z_G. Since the converse is true, the lemma follows. □
Remark. – This lemma does not depend on the field F and hence is true when G is a group over a finite field.

Définition 3 *Let* V *be a smooth irreducible algebraic variety over* F*. A Lebesgue measure on* V(F) *is a measure whose image in any analytic map of* V(F) *is comparable to the standard Lebesgue measure on the affine space.*

Lemma 6.3 *Let* V *be a smooth irreducible algebraic variety over* F*,* W *a proper Zariki closed subset of* V *and* μ *a Lebesgue measure on* V(F)*. Then* $\mu(W(F)) = 0$*.*

Démonstration : If V has dimension 0, the assertion is clear. We now proceed by induction on the dimension of V. As V is smooth we can cover it with a finite number of Zariski open sets U together with étale morphisms $\pi : U \to F^n$. Hence, by Fubini theorem, we can suppose V is an affine space. We can then assume W to be an hypersurface given by an equation

$$A_0(X_1, \dots, X_{n-1}) + A_1(X_1, \dots, X_{n-1})X_n + \cdots + A_d(X_1, \dots, X_{n-1})X_n^d = 0$$

with $d \geq 1$ and $A_d \neq 0$. Let W_0 be the vanishing locus of A_d in F^{n-1}. By induction hypothesis $W_0(F)$ has zero Lebesgue measure in F^{n-1}, hence so has $W_0(F) \times F$ in F^n. Moreover Fubini theorem also implies that $W(F) \setminus (W_0(F) \times F)$ has zero Lebesgue measure.
□

Lemma 6.4 *For* γ *in* G(F)*, let* Y *be a closed proper Zariski subset of* $X = G_\gamma \backslash G$*. Viewing* $Z = G_\gamma(F) \backslash G(F)$ *as a subset of* X(F)*,* $Y(F) \cap Z$ *has 0 measure for any* G(F)*-invariant measure on* Z*.*

Démonstration : By construction of the quotient measure, any G(F)-invariant measure on Z is the restriction of a Lebesgue measure on X(F). Hence the assertion is a corollary of 6.3. □

Corollary 6.5 *Let* P *be an essential parabolic subgroup of* G*. If* γ *is a non-central rational element of* G*, then* $\mathcal{O}_\gamma^{G(F)}(1_{P(F)}) = 0$*.*

Démonstration : Since γ is not central, by lemma 6.2 P intersects the orbit of γ along a proper closed subset. In other words $P_\gamma \backslash P$ is a closed proper Zariski subset of $G_\gamma \backslash G$. Whence the assertion, thanks to lemma 6.4. □

Corollary 6.6 *Let* $P = MN$ *and* $P' = M'N'$ *be two proper rational parabolic subgroups of* G *together with rational Levi decomposition and* K *a maximal compact subgroup of* G(F) *such that* $G(F) = KP'(F)$*. Then for any* γ *in* M'(F)*,* $(1_{P(F)})_{M'}(\gamma) = 0$*.*

Démonstration : By definition

$$(1_{\mathrm{P(F)}})_{\mathrm{M}'}(\gamma) = \int_{\mathrm{K}} \int_{\mathrm{N}'(\mathrm{F})} \int_{\mathrm{M}'_\gamma(\mathrm{F})\backslash \mathrm{M}'(\mathrm{F})} 1_{\mathrm{P(F)}}(k^{-1}m^{-1}\gamma mnk) dk.dn.dm$$

so that for the sum over $\mathrm{N}'(\mathrm{F})$ not to vanish there must exist some n_0 in $\mathrm{N}'(\mathrm{F})$ such that $m^{-1}\gamma mn_0$ belongs to $k\mathrm{P}(F)k^{-1}$. Suppose k and m fixed and n_0 having the preceding property, then

$$\forall n \in \mathrm{N}'(\mathrm{F}) \qquad m^{-1}\gamma mn \in k\mathrm{P}(F)k^{-1} \Leftrightarrow n_0^{-1}n \in k\mathrm{P}(F)k^{-1} \ .$$

Now the set of (n, g) in $\mathrm{N}' \times \mathrm{G}$ such that $g^{-1}n_0^{-1}ng$ belongs to P is a closed proper Zariski subset of $\mathrm{N}' \times \mathrm{G}$, hence by lemma 6.4, the set of (n, g) in $\mathrm{N}'(\mathrm{F}) \times \mathrm{G}(\mathrm{F})$ such that $g^{-1}n_0^{-1}ng$ belongs to P(F) has zero measure. Hence, by Iwasawa decomposition the set of (n, k) in $\mathrm{N}'(\mathrm{F}) \times \mathrm{K}$ such that $k^{-1}n_0^{-1}nk$ belongs to P(F) also has zero measure. Whence the assertion. □

7 Vanishing families and parahoric subgroups

7.1 We keep the assumptions of the preceding section and assume F to have non-zero residual characteristic p, except when noted. We write $\mathfrak{o}_\mathrm{F}$ for the ring of integers of F, $\mathfrak{p}_\mathrm{F}$ for the maximal ideal in $\mathfrak{o}_\mathrm{F}$ and set $q = |\mathfrak{o}_\mathrm{F}/\mathfrak{p}_\mathrm{F}|$. Let P_0 be a minimal rational parabolic subgroup of G, with rational Levi subgroup M_0, and W^0_G the Weyl group of G relative to M_0. If P is a rational parabolic containing P_0, let M_P be a rational Levi subgroup, common to P and to its opposed parabolic subgroup, and A_P the maximal split torus contained in its center.

Lemma 7.2 *Let* $\mathrm{P} = \mathrm{MN}$ *be a rational parabolic subgroup of* G *containing* P_0 *together with a rational Levi decomposition. If* $\gamma = m_0n_0$ *belongs to* P *with* $m_0 \in \mathrm{M}$ *and* $n_0 \in \mathrm{N}$ *and if* w *is an element of* W^0_G*, then*

$$\mathrm{X} = \{n \in \mathrm{N} \mid Ad(m_0)(n^{-1})n_0n \in w\mathrm{P}w^{-1}\}$$

is a subvariety of N *defined by equations with degree bounded by a constant* $d(\mathrm{G})$ *depending only on the root system of* G*. In particular* X *is contained in an hypersurface of* N *of degree* $d(\mathrm{G})$*.*

Démonstration : Fix isomorphisms θ_α of $\mathbf{G}_a$ with U_α. By [Bor91, 14.5 (*)], the commutator $\theta_\alpha(-x)\theta_\beta(-y)\theta_\alpha(x)\theta_\beta(y)$ is a product of some $\theta_{\alpha_i}(P_i(x, y))$'s for α_i of the form $r\alpha + s\beta$ with r and s positive integers and P_i a monomial of degree r in x and s in y. In particular the total degree of P_i is bounded by a constant d_0 depending only on the root system of G.

Let n be in N. Write n and n_0 as products of $\theta_\alpha(x_\alpha)$'s and $\theta_\alpha(y_\alpha)$'s for a arbitrary fixed order on the roots α's. Then n_0n is a product (in the same order) of $\theta_\alpha(z_\alpha)$'s with z_α's polynomials in x_α's and y_α's of total degree bounded by $d_0(d - 1)$, where d is the number of α's involved. In particular d is bounded by a constant depending only on the root system of G.

Now let's write m_0 as n^-tn^+ with t in T and n^- and n^+ products of $\theta_\alpha(u_\alpha)$'s with α negative, for n^-, and α positive, for n^+. Hence $Ad(m_0)(n^{-1})$ is a product of $\theta_\alpha(v_\alpha)$'s for v_α polynomials in u_α (with coefficients depending on t) with total degree bounded by $d_0 d'$, with d' bounded by a constant depending only on the root system of G.

Hence $Ad(m_0)(n^{-1})n_0 n$ depends polynomialy on x_α's, the polynomial being of total degree bounded by a constant depending only on the root system of G. □

Lemma 7.3 *Let* A *be an affine space over the finite field* $\mathbf{F}_q$ *and* Σ *be an hypersurface of* A *of degree* d*. Then*

$$\mathrm{Card}\,\Sigma \leq \frac{d}{q}\,\mathrm{Card}\,\mathrm{A}\ .$$

Démonstration : This follows from the fact that a polynomial on n variables over $\mathbf{F}_q$ and total degree d has at most dq^{n-1} zeroes. □

Corollary 7.4 *Let* K *be an hyperspecial compact subgroup of* G(F) *and* P *a parahoric subgroup of* K*. Write* $\overline{\mathrm{P}}$ *for the parabolic subgroup of* $\overline{\mathrm{G}} = \mathrm{G}(\mathfrak{o}_\mathrm{F}/\mathfrak{p}_\mathrm{F})$ *obtained by reduction modulo* $\mathfrak{p}_\mathrm{F}$ *and suppose* $\overline{\mathrm{P}}$ *is an essential parabolic subgroup of* $\overline{\mathrm{G}}$*. For* γ *in* K *non central modulo* $\mathfrak{p}_\mathrm{F}$*, there exists a constant* $c_1(\mathrm{G})$*, depending only on the root system of* G*, such that*

$$\mathcal{O}_\gamma^{\mathrm{G(F)}}(1_\mathrm{P}) \leq \frac{c_1(\mathrm{G})}{q}\mathcal{O}_\gamma^{\mathrm{G(F)}}(1_\mathrm{K})\ .$$

Démonstration : We may suppose P to be a maximal proper parahoric subgroup of K.

Let's split the G(F)-orbit of γ in K-orbits :

$$\mathcal{O}_\gamma^{\mathrm{G(F)}} = \cup_i \mathcal{O}_{\gamma_i}^{\mathrm{K}}\ .$$

We may suppose each γ_i either in P or such that its K-orbit does not meet P.

Let π be the reduction morphism from K to $\overline{\mathrm{G}}$. For g in K, let $\bar{g} = \pi(g)$ and, for $\bar{g}$ in $\overline{\mathrm{G}}$, let $[\bar{g}]$ be the fiber of π above $\bar{g}$. Then

$$\mathcal{O}_{\gamma_i}^{\mathrm{K}} = \cup_{\bar{g}\in\mathcal{O}_{\bar{\gamma}_i}^{\overline{\mathrm{G}}}} \mathcal{O}_{\gamma_i}^{\mathrm{K}} \cap [\bar{g}]\mathcal{O}_{\gamma_i}^{\mathrm{K}} \cap \mathrm{P} = \cup_{\bar{g}\in\mathcal{O}_{\bar{\gamma}_i}^{\overline{\mathrm{G}}}\cap\overline{\mathrm{P}}} \mathcal{O}_{\gamma_i}^{\mathrm{K}} \cap [\bar{g}]$$

and, for k in K the map $Ad(k)$ induces an isomorphism

$$\mathcal{O}_{\gamma_i}^{\mathrm{K}} \cap [\bar{g}] \to \mathcal{O}_{\gamma_i}^{\mathrm{K}} \cap [\bar{k}^{-1}\bar{g}\bar{k}]$$

between fibers. Hence

$$\mathrm{vol}_{\mathcal{O}_\gamma^{\mathrm{G(F)}}}(\mathcal{O}_{\gamma_i}^{\mathrm{K}}\cap\mathrm{K}) = |\mathcal{O}_{\bar{\gamma}_i}^{\overline{\mathrm{G}}}|\mathrm{vol}_{\mathcal{O}_\gamma^{\mathrm{G(F)}}}(\mathcal{O}_{\gamma_i}^{\mathrm{K}}\cap[\bar{\gamma}_i])\mathrm{vol}_{\mathcal{O}_\gamma^{\mathrm{G(F)}}}(\mathcal{O}_{\gamma_i}^{\mathrm{K}}\cap\mathrm{P}) = |\mathcal{O}_{\bar{\gamma}_i}^{\overline{\mathrm{G}}}\cap\overline{\mathrm{P}}|\mathrm{vol}_{\mathcal{O}_\gamma^{\mathrm{G(F)}}}(\mathcal{O}_{\gamma_i}^{\mathrm{K}}\cap[\bar{\gamma}_i])\ .$$

Hence we're left to prove

$$|\mathcal{O}_{\bar{\gamma}_i}^{\overline{\mathrm{G}}} \cap \overline{\mathrm{P}}| \leq \frac{c_1(\mathrm{G})}{q}|\mathcal{O}_{\bar{\gamma}_i}^{\overline{\mathrm{G}}}|$$

which amounts to

$$|\{\bar{g} \in \overline{\mathrm{G}} \mid \bar{g}^{-1}\bar{\gamma}_i\bar{g} \in \overline{\mathrm{P}}\}| \leq \frac{c_1(\mathrm{G})}{q}|\overline{\mathrm{G}}|\ .$$

By Bruhat decomposition, $\overline{\mathrm{G}}$ is a disjoint union :

$$\overline{\mathrm{G}} = \overline{\mathrm{P}} \cup \overline{\mathrm{N}} w \overline{\mathrm{P}}$$

where $\overline{\mathrm{N}}$ is the unipotent radical of $\overline{\mathrm{P}}$ and w the unique non trivial element of the Weyl group of $\mathrm{A_P}$ in G. If $\bar{g}$ belongs to $\overline{\mathrm{P}}$, then so is $\bar{g}^{-1}\bar{\gamma}_i\bar{g}$ since we chose γ_i in P. Moreover

$$|\overline{\mathrm{G}}|/|\overline{\mathrm{P}}| = 1 + |w\overline{\mathrm{N}}/w\overline{\mathrm{N}} \cap \overline{\mathrm{P}}|$$

whence

$$|\overline{\mathrm{P}}| \leq \frac{1}{q+1}|\overline{\mathrm{G}}| \ .$$

Hence it suffices to show

$$|\{nwp \in \overline{\mathrm{N}}w\overline{\mathrm{P}} \mid (nw)^{-1}\bar{\gamma}_i(nw) \in \overline{\mathrm{P}}\}| \leq \frac{c}{q}|\overline{\mathrm{N}}w\overline{\mathrm{P}}| \ \left(\leq \frac{c}{q}|\overline{\mathrm{G}}|\right) \ ,$$

that is

$$|\{(n,p) \in \overline{\mathrm{N}} \times \overline{\mathrm{P}} \mid (nw)^{-1}\bar{\gamma}_i(nw) \in \overline{\mathrm{P}}\}| \leq \frac{c}{q}|\overline{\mathrm{N}}|.|\overline{\mathrm{P}}|$$

or

$$|\{n \in \overline{\mathrm{N}} \mid (nw)^{-1}\bar{\gamma}_i(nw) \in \overline{\mathrm{P}}\}| \leq \frac{c}{q}|\overline{\mathrm{N}}| \ .$$

By lemma 7.2, the left hand side is the set of points over $\mathfrak{o}_\mathrm{F}/\mathfrak{p}_\mathrm{F}$ of a Zariski closed subset of F given by equations of degree bounded by $d(\mathrm{G})$. Moreover, by lemma 6.2, it is a proper subset, hence is included in an hypersurface of degree at most $d(\mathrm{G})$.

Since N is isomorphic, as a variety, to an affine space, the assertion results from lemma 7.3. □

Corollary 7.5 *Let* $\mathrm{P}' = \mathrm{M}'\mathrm{N}'$ *be a proper rational parabolic subgroup of* G *together with a rational Levi decomposition,* K *a hyperspecial compact subgroup of* G(F) *and* P *a parahoric subgroup of* K *whose reduction is essential. There exists a constant* $c_2(\mathrm{G})$ *depending only on* G *such that, for any* γ *in* M′(F)

$$\int_\mathrm{K}\int_{\mathrm{N}'(\mathrm{F})}\int_{\mathrm{M}'_\gamma(\mathrm{F})\backslash \mathrm{M}'(\mathrm{F})} \left(\frac{c_2(\mathrm{G})}{q}1_\mathrm{K} - 1_\mathrm{P}\right)(k^{-1}m^{-1}\gamma mnk)dk.dm.dn \geq 0 \ .$$

Démonstration : The proof proceeds as in corollary 6.6. For the sum over N′(F) not to vanish, there must exist n_0 in N′(F) such that $m^{-1}\gamma m n_0$ belongs to K. By change of variable, we may suppose $n_0 = 1$, hence by reduction we're left to prove that the number of (k,n) in $\overline{\mathrm{G}} \times \overline{\mathrm{N}}'$ such that $k^{-1}nk$ belongs to $\overline{\mathrm{P}}$ is bounded by $c_2(\mathrm{G})/q$ times the cardinality of $\overline{\mathrm{G}} \times \overline{\mathrm{N}}'$.

Let us first remark that the set of (n,g) in $\mathrm{N}' \times \mathrm{G}$ such that $g^{-1}ng$ belongs to P is a proper Zariski closed subset of $\mathrm{N}' \times \mathrm{G}$ and hence its number of points over $\mathfrak{o}_\mathrm{F}/\mathfrak{p}_\mathrm{F}$ is bounded by a constant (depending on N′, P and G, hence only on G) times q to the power the dimension of $\mathrm{N}' \times \mathrm{G}$ minus one.

By Bruhat decomposition the number of points of $\mathrm{N}' \times \mathrm{G}$ over $\mathfrak{o}_\mathrm{F}/\mathfrak{p}_\mathrm{F}$ is bounded from below by a constant times q to the power its dimension. Whence the assertion. □

8 Annexe – Borne de Varshamov-Gilbert

Pour étudier les différents codes de façon asymptotique, on introduit V_q l'ensemble des points (x, R) pour C variant (et q fixé) et U_q son adhérence. Le théorème fondamental de cette théorie est le suivant

Théorème 2 *Il existe une fonction continue α_q de $[0,1]$ dans lui-même telle que U_q soit l'ensemble des points (x, R) vérifiant $0 \leq R \leq \alpha_q(x)$. On a*

$$1 - x\log_q(q-1) + x\log_q(x) - (1-x)\log_q(1-x) \leq \alpha_q(x) \leq \max\{1 - \frac{q}{q-1}x; 0\} .$$

La fonction minorante (souvent notée β_q s'appelle la borne de Varshamov-Gilbert.

Références

[Art89] James Arthur, *The L^2-Lefschetz numbers of Hecke operators*, Inventiones Mathematicæ 97 (1989), 257–290.

[ACl89] James Arthur, Laurent Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Annals of mathematics studies, Princeton University Press (1989).

[Bom73] Enrico Bombieri, *Counting points on curves over finite fields*, Séminaire Bourbaki 430, 1973.

[Bor91] Armand Borel, *Linear algebraic groups*, Graduate texts in mathematics 126, Springer-Verlag, 1991.

[Bou81] Nicolas Bourbaki, *Groupes et algèbres de Lie*, Chapitres 4, 5 et 6, Masson, 1981.

[Car79] Pierre Cartier, *Representations of p-adic groups : a survey*, in *Automorphic forms, representations and L-functions*, Proceedings of simposia in pure mathematics volume XXXIII, part 1 (1979), 111–155.

[Del71] Pierre Deligne, *Travaux de Shimura*, Séminaire Bourbaki 389, Février 1971, 1–40.

[Dri74] V. G. Drinfeld, *Elliptic modules*, Mat. Sb. 94 (136), N4, 1974, 594–627.

[DVl83] Vladimir G. Drinfeld, Serge G. Vlăduţ, *Number of points of an algebraic curve*, Funktsionalnyi Analiz i ego Prilozheniya 17 (1983), 68–69.

[Elk96] N. Elkies, *Interprétation de Garcia-Stichtenoth en terme des "courbes modulaires de Drinfeld"*, non publié.

[GSt95] Arnaldo Garcia, Henning Stichtenoth, *A tower of Artin-Schreier extensions of function fields attaining the Drinfeld-Vladut bound*, Inventiones Mathematicæ 121 (1995), 211–222.

[GKM97] Marc Goresky, Robert E. Kottwitz, Robert D. McPherson, *Discrete series characters and the Lefschetz formula for Hecke operators*, Duke Math Journal 89 (1997), 477–554.

[GKM98] Marc Goresky, Robert E. Kottwitz, Robert D. McPherson, *Correction to "Discrete series characters and the Lefschetz formula for Hecke operators"*, Duke Math Journal 92 (1998), 665–666.

[Iha82] Yasutaka Ihara, *Some remarks on the number of rational points of algebraic curves over finite fields*, Journal of the faculty of science of Tokyo 28, 1981, 721–724.

[Kot84] Robert E. Kottwitz, *Shimura varieties ans twisted orbital integrals*, Mathematische Annalen 269 (1984), 287–300.

[Kot86] Robert E. Kottwitz, *Stable trace formula : elliptic singular terms*, Mathematische Annalen 275 (1984), 365–399.

[Kot88] Robert E. Kottwitz, *Shimura varieties and λ-adic representations*, in Automorphic forms, Shimura varieties and L-functions, Proceedings of a conference held at the University of Michigan, Ann Arbor, July 6–16, 1988, volume I, 161–209.

[Lan80] Robert P. Langlands, *Base change for $GL(2)$*, Annals of mathematics studies 96, Princeton University Press, 1980.

[Man81] Yuri I. Manin, *What is the maximum number of points on a curve over* $\mathbf{F}_2$ *?*, Journal of the faculty of science of Tokyo 28, 1981, 715–720.

[Mil92] James S. Milne, *The points on a Shimura variety modulo a prime of good reduction*, in *The Zeta functions of Picard modular surfaces*, Robert P. Langlands and Dinakar Ramakrishnan editors, Les publications CRM, 1992, 151–253.

[NXi98] Harald Niederreiter, Chaoping Xing, *Towers of function fields with asymptotically many rational places and an improvement of the Gilbert-Varshamov bound*, Mathematische Nachrichten 195 (1998), 171–186.

[Rei97] Harry Reimann, *The semi-simple zeta function of quaternionic Shimura varieties*, Lecture notes in mathematics 1657, 1997.

[RZi91] Harry Reimann et Thomas Zink, *The good reduction of Shimura varieties associated to quaternion algebras over a totally real number field*, Preprint, University of Toronto, 1991.

[Ser83] Jean-Pierre Serre, *Sur le nombre de points rationnels d'une courbe algébrique sur un corps fini*, Comptes rendus de l'académie des sciences de Paris 296, 1983, 397–402.

[Ser97] Jean-Pierre Serre, *Répartition asymptotique des valeurs propres de l'opérateur de Hecke T_p*, Journal of American Mathematical Society 10 (1997), 75–102.

[Tsf92] Michael A. Tsfasman, *Some remarks on the asymptotic number of points*, in Lecture Notes in Mathematics 1518, 1992, 178-192.

[TVl96] Michael A. Tsfasman et S. G. Vlăduţ, Asymptotic properties of zeta functions, Prépublications de l'I.M.L., numéro 96-12, CNRS, Marseille, 1996.

[TVZ82] Michael A. Tsfasman, S. G. Vlăduţ et Thomas Zink, *Modular curves, Shimura curves and Goppa codes, better than Varshamov-Gilbert bound*, Mathematische Nachrichten 109, 1982, 21–28.

[Zin85] Thomas Zink, *Degeneration of Shimura surfaces and a problem in coding theory*, in Fundamentals of computation theory, Cottbus, Lectures Notes in Computer Science 199, Springer-Verlag (1985), 503–511.