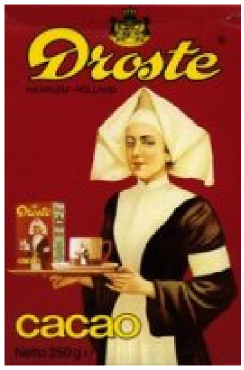
# L'effet Droste




Le 24 août 2009, par **Jos Leys**
Mathematical Imagery (**page web**)

***La relation entre M.C. Escher, le chocolat, et les nombres complexes.***

EN 1956, *Maurits Cornelis Escher* compléta un dessin qu'il appelait *Prententententoonstelling* , « Galerie d'images ». Le dessin montre un jeune homme qui regarde une image déformée d'une manière qui semble ne pas avoir de sens. Au milieu de l'image, on voit une tache blanche énigmatique.

En 2003, une équipe de mathématiciens de l'université de Leiden, sous la direction du Professeur Hendrik Lenstra a réussi à déchiffrer la structure mathématique de l'image. Une fois cette structure connue, ils ont pu « compléter » l'image en remplissant la tache blanche, à l'aide d'un algorithme numérique.

Ils ont publié **un article** là-dessus qui a été particulièrement bien accueilli aussi bien dans des cercles universitaires que dans la presse générale [1]. À la surprise générale, l'équipe de Lenstra a montré que la structure de l'image est la même que la structure d'une image à *effet Droste*. L'*effet Droste* tire son nom d'un paquet de cacao de la compagnie du ***chocolatier Droste*** aux Pays-Bas qui montre une infirmière portant un plateau sur lequel il y a un paquet de cacao. L'image sur ce paquet de cacao est encore une infirmière portant un plateau sur lequel il y a un paquet de cacao... (*ad infinitum*) [2]

Comment construire une copie d'une image quelque part dans l'image et comment répéter cela à l'infini ? La méthode de Lenstra pour la transformation d'images est en trois étapes que nous allons expliquer [3] :

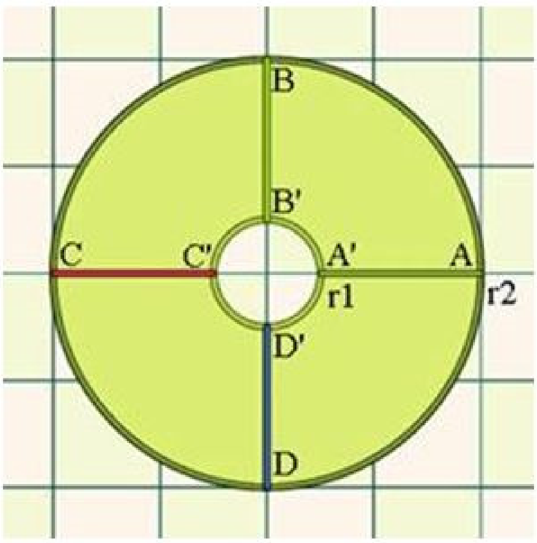
**Étape 1 :** Le logarithme.

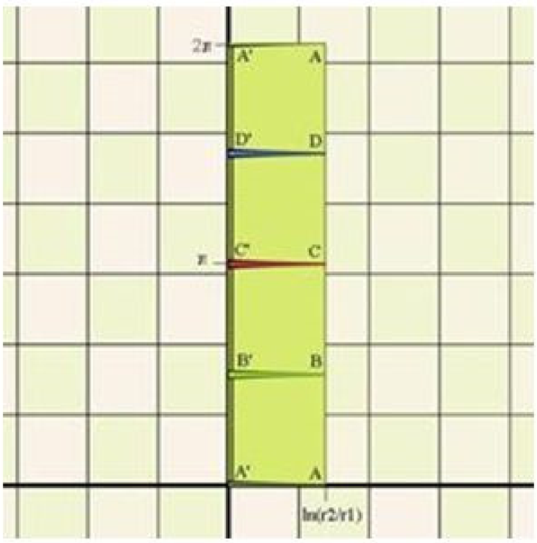
Considérer le logarithme d'un nombre complexe est une chose un peu compliquée. Un nombre complexe $z$ s'écrit souvent sous la forme $z = re^{i\theta}$ où $r \geq 0$ désigne le *module*, et $\theta$ l'*argument*. On définit alors naturellement $\ln(z)$ comme $\ln r + i\theta$ mais cela pose deux problèmes. Le premier, comme d'habitude, est qu'il faut supposer que $z$ n'est pas nul, c'est-à-dire que $r > 0$. Le second est que $\theta$ n'est pas unique : $re^{i\theta} = re^{i(\theta+2\pi n)}$ pour chaque entier $n$, si bien qu'il faut choisir un $\theta$. La tradition est de choisir celui pour lequel $0 \leq \theta < 2\pi$, mais ce n'est qu'une convention ; on pourrait en prendre une autre.

Comme $0 \leq \theta < 2\pi$, la transformation $z \mapsto \ln(z)$ transforme le plan complexe (privé de l'origine) en un ruban de hauteur $2\pi$ au-dessus de l'axe réel.

Prendre d'autres conventions pour le logarithme reviendrait à ajouter $2ni\pi$ si bien que l'image du logarithme recouvrirait alors une autre bande, translatée de la première. On obtient donc une infinité d'images du plan, suivant les conventions choisies : ceci sera important pour la suite.

Regardons comment cette transformation agit sur deux cercles concentriques (centrés à l'origine), de rayons $r_1$ et $r_2$.

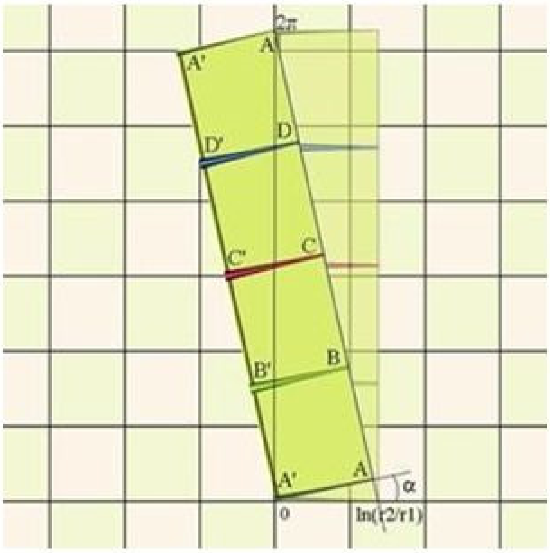


La figure illustre la transformation $z \mapsto \ln(\frac{z}{r_1})$. Le cercle intérieur devient un segment vertical passant par l'origine et le cercle extérieur devient un autre segment vertical à distance $\ln(r_2/r_1)$ du premier. C'est comme si nous avions coupé la figure le long du segment $A - A'$ et déplié le tout en un rectangle.

**Étape 2 :** rotation et dilatation.

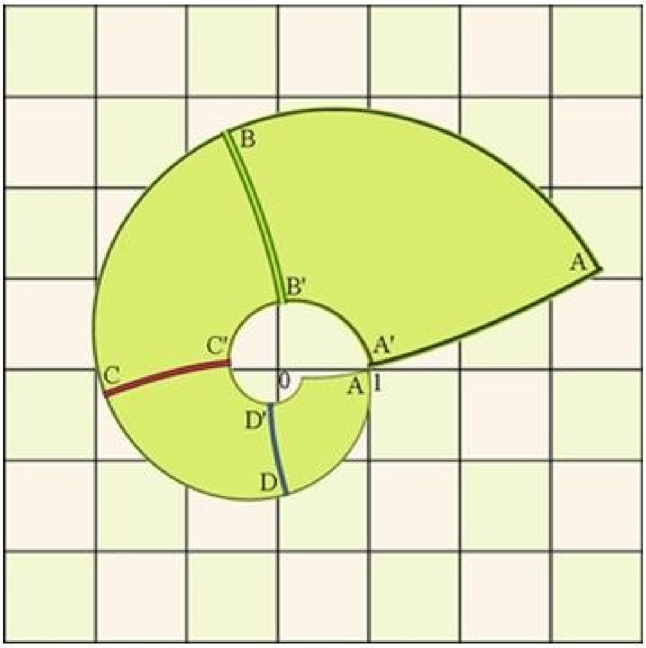
On tourne maintenant le rectangle dans la figure de droite jusqu'à ce que sa diagonale coïncide avec l'axe imaginaire, et on rétrécit le rectangle, par une homothétie, pour que la longueur de sa diagonale soit $2\pi$.




Dans la prochaine étape, on va effectuer la transformation inverse du logarithme.

**Étape 3 :** exponentielle.

On obtient l'image ci-dessous en appliquant la transformation $z \mapsto e^z$ sur le rectangle penché. Tous les côtés du rectangle sont transformés en spirales. À noter que les sommets du rectangle sur la diagonale verticale (qui se trouvent à $0$ et $2i\pi$) sont tous les deux transformés au point $1$.

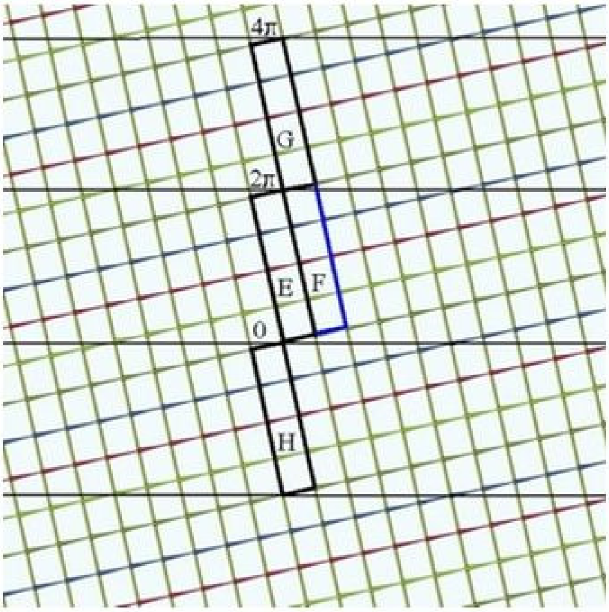


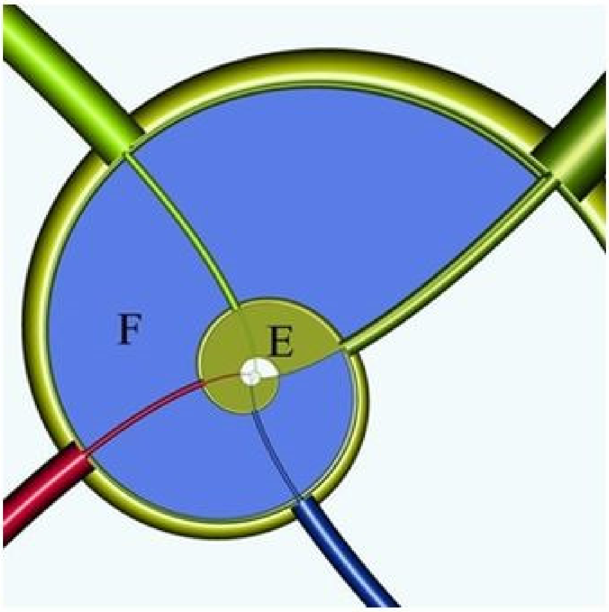

On peut calculer explicitement les transformations effectuées. Elles ne dépendent que du quotient $\frac{r_2}{r_1}$.

Lors de l'étape 2, le rectangle s'est vu appliquer la transformation $z \mapsto zfe^{i\alpha}$ avec $\alpha = \arctan(\frac{\ln(\frac{r_2}{r_1})}{2\pi})$ et $f = \cos(\alpha)$.

La composition des trois étapes donne la transformation $z \mapsto (\frac{z}{r_1})^{\beta}$, avec $\beta = fe^{i\alpha}$.

On reprend maintenant l'étape 2. On construit un pavage du plan avec le rectangle qu'on avait tourné d'un angle $\alpha$ et on applique alors la transformation de l'étape 3.

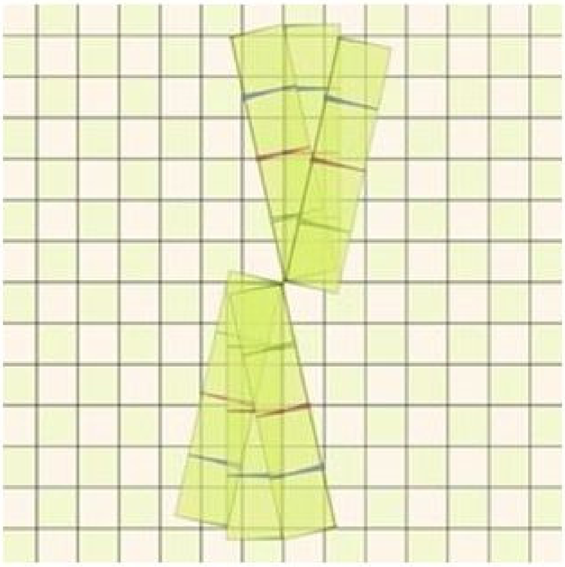



Le rectangle $E$ dans la figure de gauche est le rectangle de base et celui-ci est transformé en le domaine $E$ de la figure de droite. Le rectangle $F$ à son tour est transformé en le domaine $F$ à droite. On obtient ainsi un nombre infini de copies de l'image originale, l'anneau entre deux cercles, qui pavent le plan (toujours privé de l'origine). Tous ces pavés sont semblables : on peut passer de $E$ à $F$ sur l'image de droite par une similitude, composition d'une dilatation (d'un facteur $\left|\left(\frac{r_2}{r_1}\right)^\beta\right|$) et d'une rotation (d'un angle égal à l'argument de $\left(\frac{r_2}{r_1}\right)^\beta$).

Dans l'étape 2, nous avons dilaté la diagonale du rectangle puis effectué une rotation de telle sorte que la diagonale devienne verticale. Le but était que les deux extrémités de la diagonale aient la même image par l'exponentielle (puisqu'elles diffèrent de $2i\pi$). Mais on peut tout aussi bien faire tourner le rectangle du même angle dans l'autre sens et ce sont les extrémités de l'autre diagonale qui auront la même image par l'exponentielle : la spirale image tournera alors dans l'autre sens.

On peut aussi choisir de ne pas tourner du tout ! Ce sont maintenant les deux extrémités des côtés verticaux qui ont la même image par l'exponentielle. Enfin, on peut également effectuer une symétrie par rapport à l'origine. On voit toutes les configurations possibles sur la figure ci-dessous.

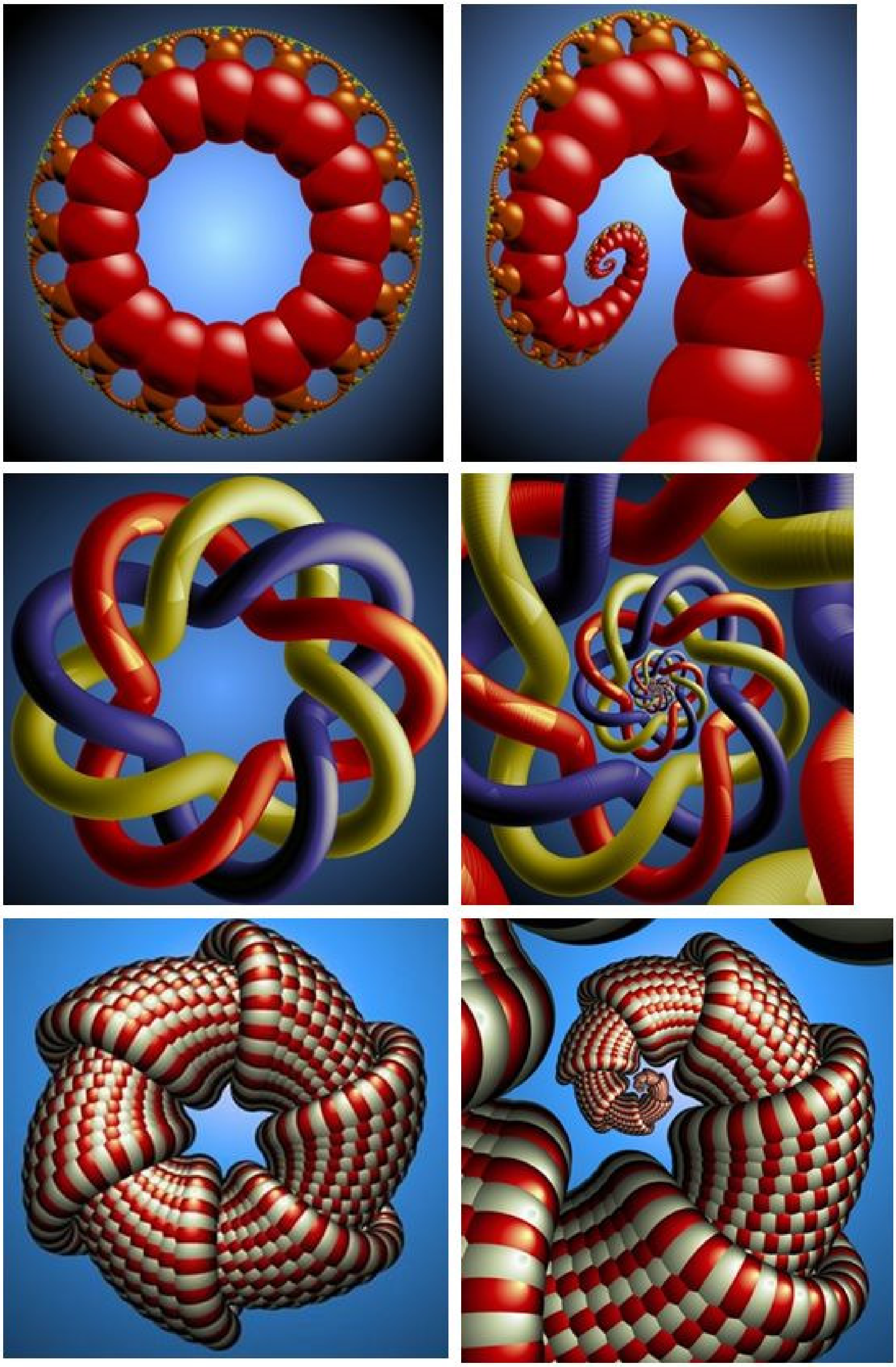


Lorsqu'on ne fait pas tourner le rectangle ou qu'on le fait tourner d'un demi-tour, la transformation va produire un nombre infini de copies concentriques de l'image originale. Dans les autres cas, il y aura une dilatation $f$ et une rotation d'angle $\alpha$, $-\alpha$, $\pi-\alpha$ ou $\pi+\alpha$, et l'image finale va produire des copies connectées en spirales.

*Récapitulons*, pour transformer une image entre deux cercles concentriques :

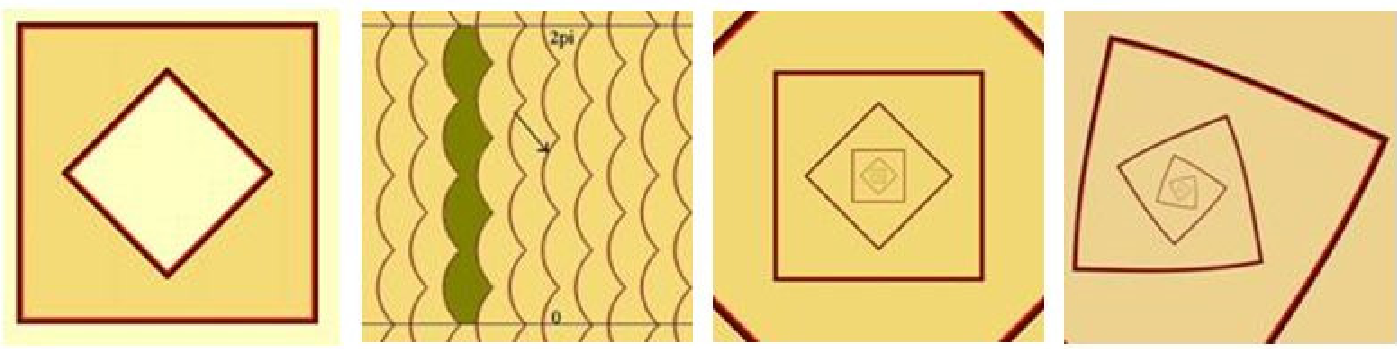
- on choisit $r_1$ et $r_2$ (une adaptation simple est nécessaire si les cercles ne sont pas centrés sur l'origine) ;
- on calcule $\alpha$, $f$ et $\beta$ ;
- on effectue le pavage avec les rectangles et on calcule $e^{\beta\ln\left(\frac{z}{r_1}\right)}$ pour tous les pixels.

Voici quelques exemples : images originales à gauche, images transformées à droite [4].

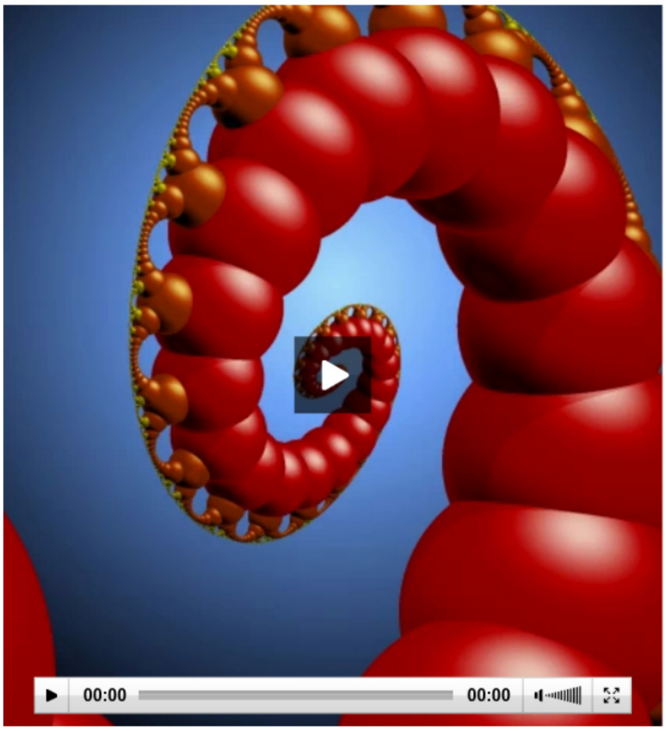


L’algorithme peut être adapté pour des cercles non concentriques ou même pour n’importe quelle forme. Prenons d’abord des carrés concentriques.

Le domaine entre les deux carrés dans l’image ci-dessous est transformé en le domaine foncé de la deuxième image par la transformation logarithme. Après exponentiation, on obtient la troisième image. Si on effectue d’abord une rotation du pavage, on obtient l’image de droite.




On peut faire des zooms illimités sur ces images. On connaît le rapport entre la taille du grand contour et celle du petit, et ceci permet de calculer la valeur de $\beta$. On peut alors calculer la dilatation et la rotation nécessaires pour obtenir une image identique. Il suffit donc de produire un petit nombre de *frames* [5] entre deux images identiques pour obtenir un film « zoom » qu'on peut alors regarder en boucle. Voici un exemple :




Les images ci-dessous montrent la situation avec deux carrés non concentriques.

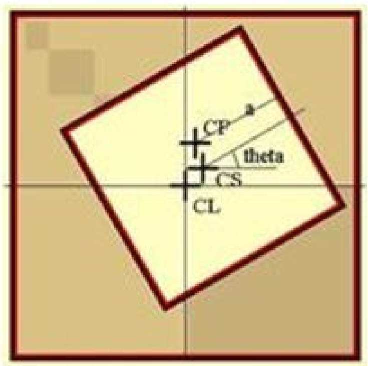
### Quelques calculs

*On a maintenant besoin du point fixe de la similitude qui transforme le grand carré en le petit. Appelons $C_L$ le centre du grand carré, $C_S$ le centre du petit carré, $\theta$ l'angle entre les deux carrés et $m$ la proportion de leur tailles. Le point fixe est alors donné par :*

$$C_F = C_L + \frac{(C_S - C_L)}{(1 - me^{i\theta})}.$$

*La transformation logarithme prend alors la forme $z \mapsto \ln(\frac{z-C_F}{a})$, où $a$ est la distance la plus courte entre $C_F$ et un coté du petit carré.*

*La transformation complète devient $z \mapsto (\frac{z-C_F}{a})^\beta$, $\beta = fe^{i\alpha}$, $f = \frac{2\pi}{(2\pi-\theta)}\cos(\alpha)$, et $\alpha = \arctan(\frac{\ln(m)}{(2\pi-\theta)})$. On emploie le même principe pour les rectangles au lieu des carrés.*





Voici un exemple de la transformation avec des rectangles non concentriques. D'abord l'image originale :

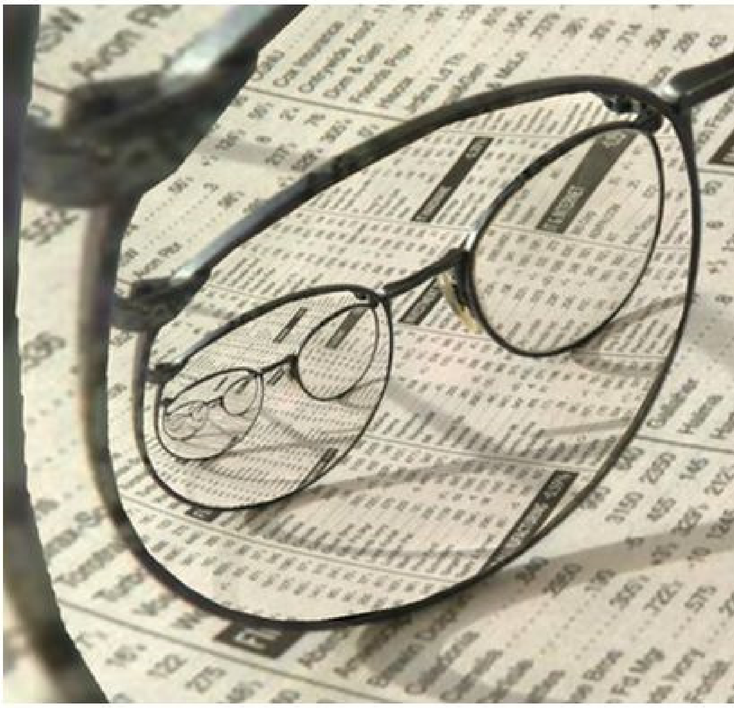
...et puis le film zoom :

On peut donc employer la transformation pour des cercles, des carrés ou des rectangles, mais il est possible aussi de l'employer pour n'importe quelle forme. Dans ce cas, il faut d'abord rendre transparente la forme qu'on veut employer dans l'image. Rendre transparente une portion d'une image numérique veut dire que la valeur des octets *alpha blending* correspondants est zéro (il faut définir la région transparente à la main à l'aide d'un logiciel graphique comme *Photoshop*). Avec une image qui a un « trou » transparent, il n'est pas trop difficile d'écrire un algorithme qui détecte le bord du trou.

Ceci est illustré par les images ci-dessous. À gauche, l'image originale [**6**]. Au milieu, une image avec section transparente, ce qui crée une région similaire (en rouge) dans l'image de droite (on peut ajouter une rotation si on le souhaite).

Selon la rotation choisie à l'étape 2 de la transformation, on obtient alors ces images :

Voici trois autres exemples d'images réalisées avec la methode du « trou transparent » [7] :



Retournons à l'image d'Escher.
L'équipe de Lenstra a montré que c'est en effet une image à effet Droste, et ils ont pu mesurer les caractéristiques : il y a une proportion $m = 256$. L'angle de rotation à l'étape 2 est donc $\alpha = \arctan(\ln(m)/2\pi) = 41.429\ldots$. degrés et le facteur $f = \cos(\alpha) = 0.749767\ldots$. Avec $\beta = fe^{i\alpha}$, on obtient $|m^\beta| = 22.5836\ldots$ et l'argument de $m^\beta$ est $157.62559\ldots$ degrés.

Ceci a donc permis de reconstruire la fameuse tache blanche. Si on fait un zoom de facteur 22.5836 et si on tourne l'image de -157.62559 degrés, on obtient une image identique. Avec cette information et l'aide d'un artiste qui a reconstruit l'image originale non déformée, le secret de la tache blanche a donc été révélé comme étant une copie de l'image originale, environ 23 fois plus petite... On peut regarder le résultat **ici**.

Il est presque incroyable que M.C. Escher, qui n'était pas mathématicien, ait pu dessiner sa « Galerie d'Images » par simple intuition. Une intuition géniale, il faut bien le reconnaître !

## Notes

[1] Regardez aussi leur **site**.

[2] Autre nom : l'effet « **Vache-qui-rit** » et aussi « **Mise en abyme** ».

[3] Cet article est une version légèrement adaptée de mon article original en anglais, publié dans *Computers and Graphics*, Volume **31**, Issue 3, June 2007.

[4] Pour d'autres exemples, regardez **cette page**.

[5] Images individuelles d'un film, normalement 25 par seconde.

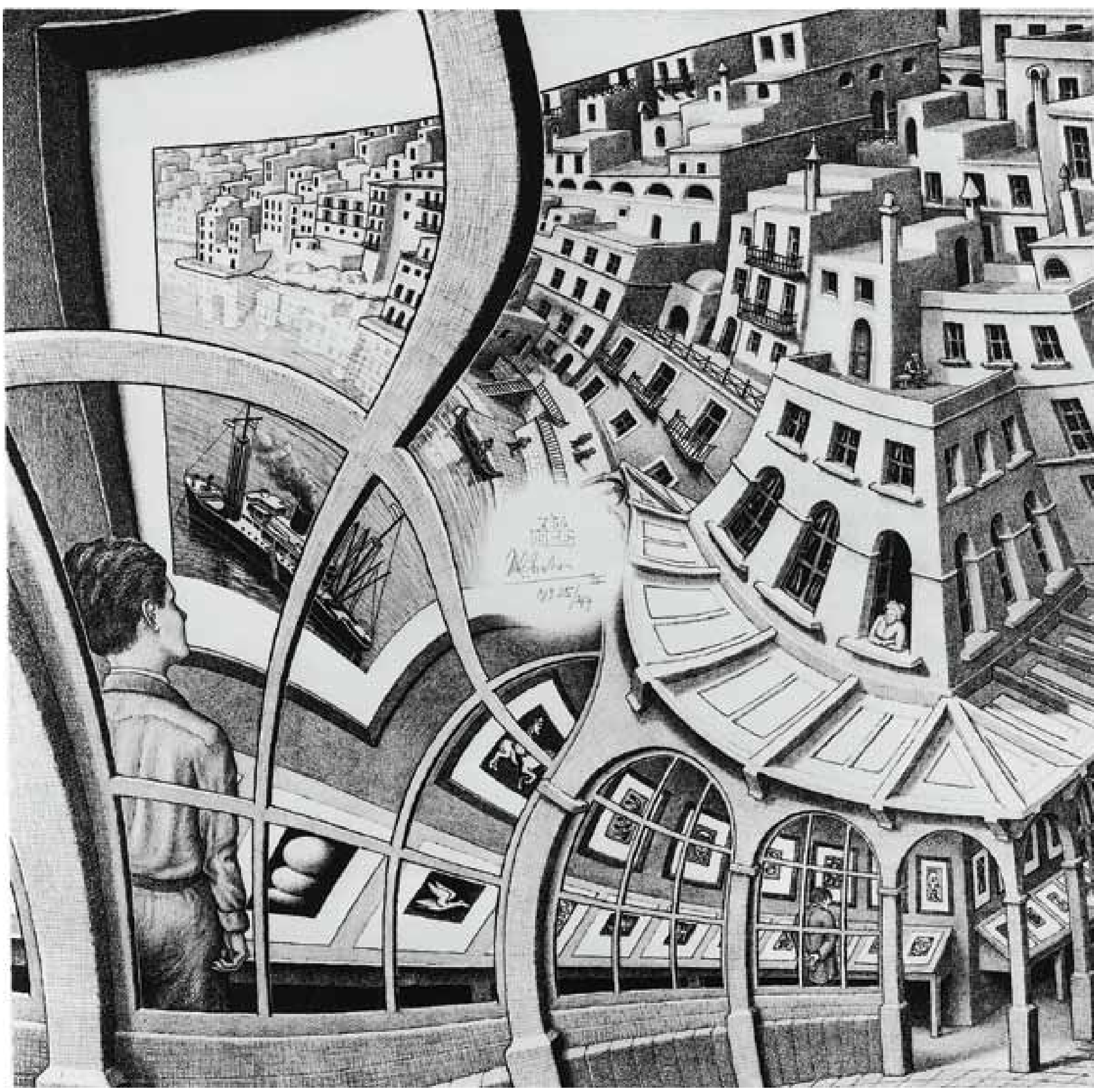
[6] Nous remercions **Étienne Ghys** d'avoir bien voulu poser !

[7] Toutes les images de cet article ont été réalisées à l'aide du logiciel **Ultrafractal**.

*Artiste fascinant, Maurits Cornelis Escher a conçu des toiles magnifiques et intrigantes, jouant avec la perspective et la géométrie, et démontrant une ingéniosité remarquable.*

# Mystérieuse lithographie d'Esc

**Philippe Carphin et Christiane Rousseau**
Université de Montréal



*Exposition d'Estampes, 1856, de M. C. Escher* 

Ses œuvres, dont plusieurs ont une renommée mondiale, sont souvent citées comme exemples d'art à saveur mathématique. Elles interpellent le spectateur par leur caractère insolite. L'une de ses œuvres, « Exposition d'Estampes » a longtemps attiré et parfois même divisé la communauté scientifique. C'est qu'elle est demeurée inachevée. Mais, était-elle vraiment inachevable ?

her

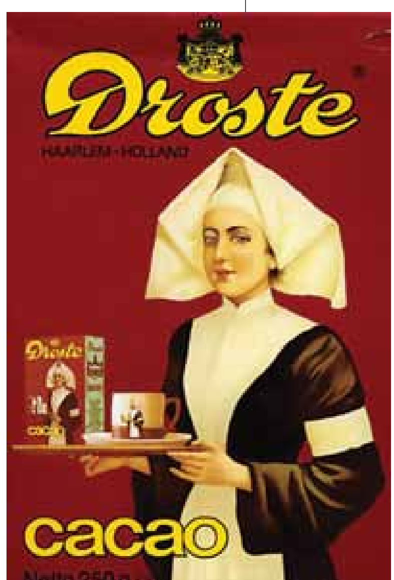
Regardons cette image, et imaginons que nous fassions un zoom pour rentrer dans l'image. En même temps nous tournons. Après avoir agrandi notre image par un facteur 22,58 et tourné dans le sens des aiguilles d'une montre d'un angle de 157,63 degrés, nous « devrions » retrouver l'image originale : devrions, parce que ce zoom nous amène sur le bord droit de la zone inachevée.

Pas très facile à voir... Pourtant, en 2003, les mathématiciens Hendrik W. Lenstra et Bart de Smit complètent la gravure ! Comment ?

*Nous allons mettre nos lunettes mathématiques pour dévoiler le mystère de cette complétion.*

Le procédé utilisé s'applique à toute image qui se retrouve reproduite à l'intérieur d'elle-même (suite à une mise en abyme) et, pour le rendre transparent, nous l'illustrerons sur une image beaucoup plus simple que la gravure d'Escher. Pour comprendre la méthode, il est recommandé de faire une lecture superficielle en se concentrant principalement sur la suite des dessins et en ignorant les détails mathématiques. Une deuxième lecture, optionnelle, permet à qui veut pousser plus loin de deviner comment ont été écrits les programmes qui génèrent ces dessins.

Hendrik Lenstra et Bart de Smit ont appelé la mise en abyme d'une image « effet Droste », à cause de l'illustration figurant sur la boîte de cacao de marque Droste vendue aux Pays-Bas. Si on décide de faire un zoom bien choisi (et une légère rotation), on retrouve la même image. On a donc une infinité de nonnes qui s'accumulent en un point. Ce point est le *centre* de l'image : une infinité de tasses s'y accumulent, une infinité de boîtes de cacao s'y accumulent, etc. En faisant des zooms successifs on voit qu'on peut, en théorie, recouvrir le plan en entier avec notre image. Si l'on fait une homothétie (un zoom) d'un certain rapport *C*, on retrouve la même image. On dit que notre image est *invariante* sous une homothétie de rapport *C*.




Le point de départ d'Escher est tout simplement une image invariante sous une homothétie de rapport 1/256. Mais, quelle transformation lui fait-il subir?

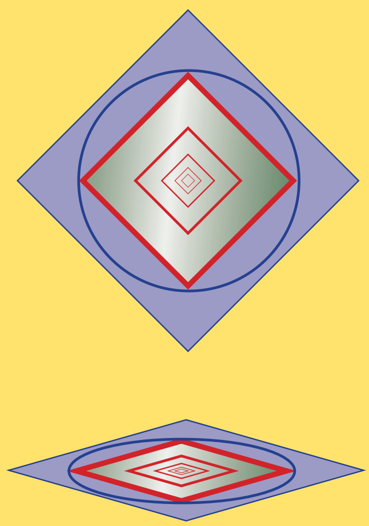
Faisons la démarche d'Escher sur l'image simple ci-contre.

Nous allons nous placer dans l'espace et imaginer notre image infinie sur le plan horizontal.

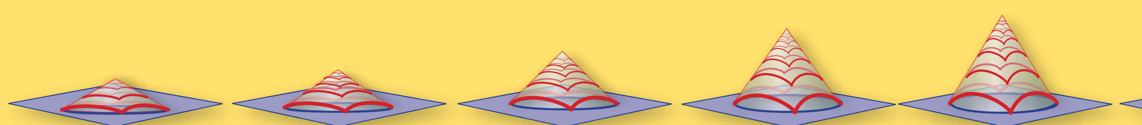
Nous supposons que cette image est élastique et nous la soulevons à partir du centre. Pendant toute cette manœuvre, nous exigeons que le cercle de rayon 1 reste fixe dans le plan horizontal. Au début nous obtenons un cône aplati, puis de plus en plus pointu. Le centre de l'image est situé au sommet du cône. Un paramètre $\alpha$ décroissant de 1 à 0 va quantifier cette manœuvre, le cône ayant une arête de longueur $1/\alpha$.

À la limite, quand le sommet du cône est à l'infini (c'est-à-dire $\alpha = 0$), le cône est devenu un cylindre. Qu'est devenue notre image? Tout au long de la déformation, on a obtenu sur le cône une image invariante sous une homothétie centrée au sommet du cône mais, au fur et à mesure qu'on étire le cône, le rapport d'homothétie se rapproche de 1. À la limite, lorsqu'on a le cylindre, il est égal à 1. Mais comme le sommet du cône est passé à l'infini, la limite de l'homothétie n'est pas l'identité mais une translation, comme

## Vue de l'image en plongée et en perspective



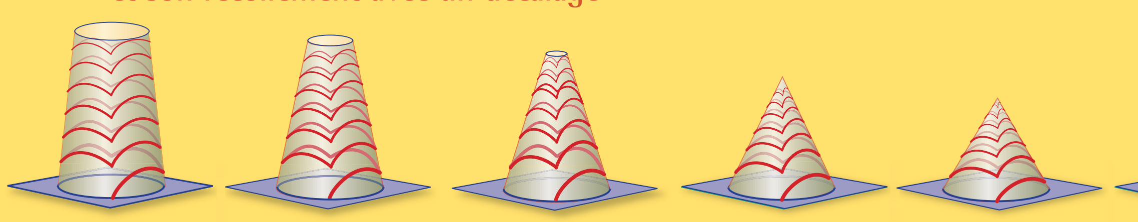
## Étirement de l'image



a) $\alpha = 9/10$ b) $\alpha = 8/10$ c $\alpha = 7/10$ d) $\alpha = 6/10$ e) $\alpha = 5/10$

Étirement du cône lorsque $\alpha$ décroît de 1 à 0. Le cercle de rayon 1 reste fixe. Le cône a une arête de longueur $\frac{1}{\alpha}$.

## Aplatissement du cône après la coupure du cylindre et son recollement avec un décalage



a) $\alpha = 1/10$ b) $\alpha = 2/10$ c) $\alpha = 3/10$ d) $\alpha = 4/10$ e) $\alpha = 5/10$

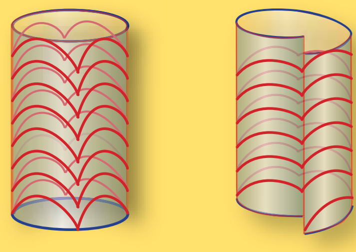
Le paramètre $\alpha$ croît de 0 à 1. Les courbes fermées sont devenues des spirales.

on le voit sur la figure. L'image sur le cylindre est invariante sous translation verticale! Elle est donc périodique avec une période verticale $T_1$. Sur le cylindre on observe une infinité d'images identiques sur des bandes l'une au-dessus de l'autre.

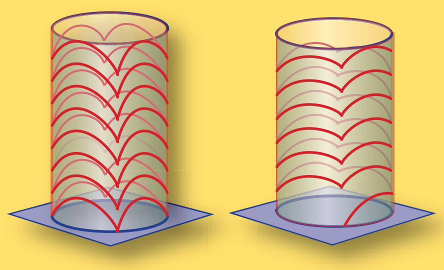
Coupons maintenant notre cylindre suivant une droite verticale. On peut faire glisser les deux côtés de la coupure l'un sur l'autre et les recoller après un décalage de $T_1$. La nouvelle image sur le cylindre est maintenant une spirale infinie, toujours périodique sous la même période verticale. Il ne reste plus qu'à faire l'opération inverse : aplatir notre cylindre en un cône jusqu'à ce que l'image finisse par être écrasée dans le plan.

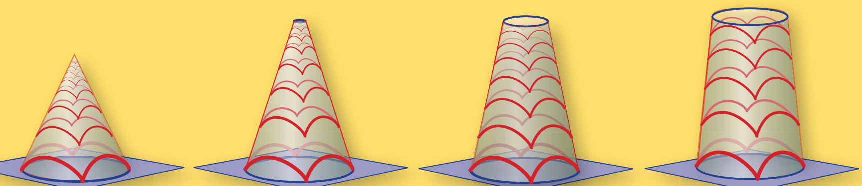
Mais comment mettre cela en équation de manière à pouvoir en programmer les étapes? On imagine que l'image est imprimée sur une feuille infinie enroulée sur le cylindre ou sur le cône et qu'on déroule cette feuille. Une feuille enroulée sur le cône aura la forme d'un secteur, mais on peut prendre l'angle du secteur arbitrairement grand.

### Cylindre coupé et recollé avec un décalage

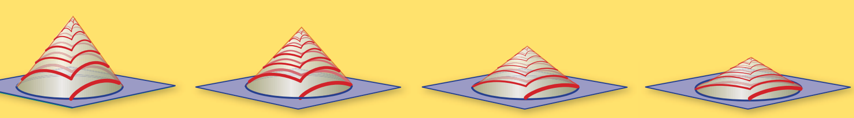


### Le cylindre coupe le plan le long du cercle de rayon 1



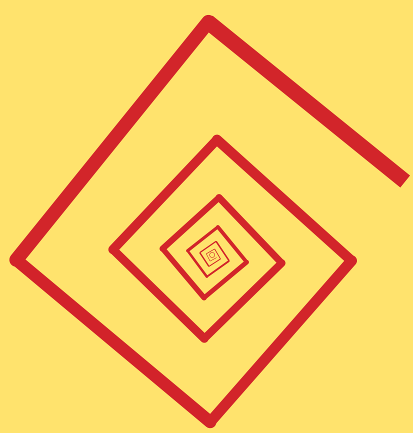


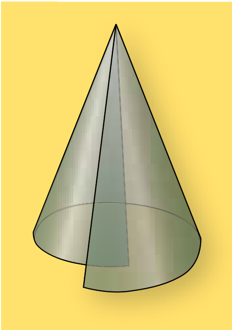
f) α = 4/10 g) α = 3/10 h) α = 2/10 i) α = 1/10



f) α = 6/10 g) α = 7/10 h) α = 8/10 i) α = 9/10

### Image finale

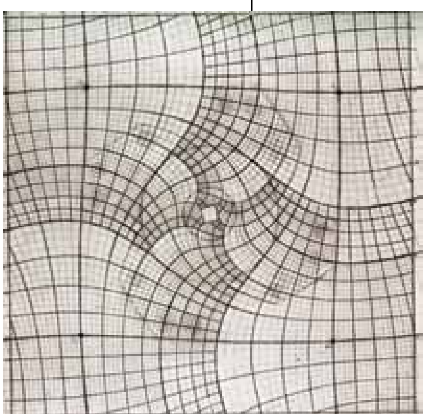




Lorsqu'on déplie une feuille enroulée sur le cône, le motif qu'on obtient ne se referme pas après un tour.

Pour notre dessin dans le plan, puisque le cercle de rayon 1 reste devant nos yeux, le sommet de la feuille s'éloigne à l'infini. Lorsqu'on déroule la feuille enroulée sur le cylindre, l'image obtenue est périodique sous deux périodes : la période $T_1$ (qu'on représente horizontalement) et une période $T_2 = 2\pi$, soit la circonférence du cercle, qu'on dessine verticalement.

Mais alors, on a aussi des périodes obliques! Voilà l'origine du fameux angle de 157,6255960832 degrés qui a tellement intrigué Hendrik Lenstra. On tourne la figure de manière à amener le vecteur $T_1 + T_2$ en position verticale. On fait une homothétie de manière à ramener sa longueur à $2\pi$.



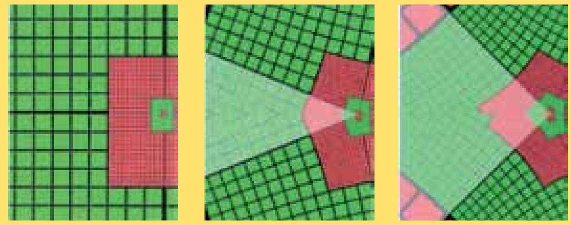
*Grille d'Escher.*

Et, on fait la transformation inverse. En faisant cela à partir d'une grille carrée, on obtient une grille semblable aux grilles de construction qu'on retrouve dans les dessins d'Escher (figure ci-contre).

Toutes les constructions d'Escher conservent les angles : ce sont des transformations *conformes*[1]. L'analyse complexe nous fournit des formules très simples pour ces transformations (voir encadré).

Vous pouvez regarder une animation de la construction d'une telle image réalisée par Philippe Carphin sur le site de la revue : www.accromath.ca.

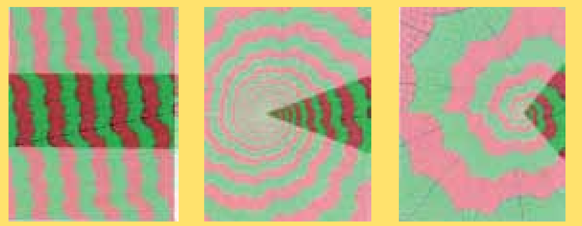
## Motif sur un cône déroulé lorsque α



*a)* $\alpha = 1$ *b)* $\alpha = 7/8$ *c)* $\alpha = 6/8$

L'image de départ est une grille du style de celle utilis

## Motif sur un cône déroulé lorsque α



*a)* $\alpha = 0.0001$ *b)* $\alpha = 1/8$ *c)* $\alpha = 2/8$

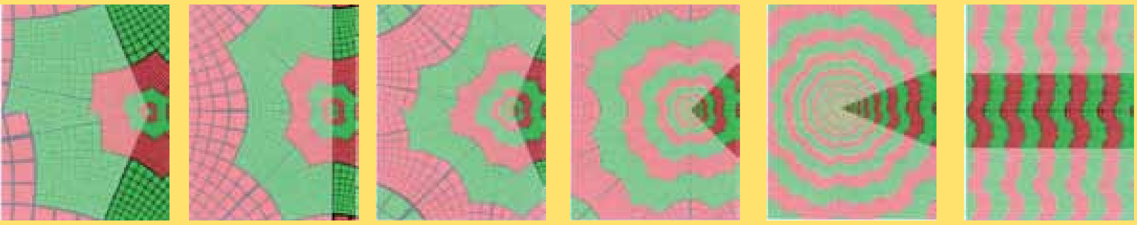
Soyez à l'affût et tâchez de trouver une occasion de regarder le documentaire de Jean Bergeron « Achever l'inachevable », dans lequel Hendrik Lenstra raconte sa fascination devant cette gravure et le « Eureka » qui a permis à son équipe d'entreprendre la longue tâche de complétion de la gravure.

1. *Voir* La cartographie, *dans Accromath vol. 3, hiver-printemps 2008.*

*Dessin complété par l'équipe de Lenstra, avant transformation.*

*Gravure complétée par l'équipe de Lenstra à partir du dessin complété.*

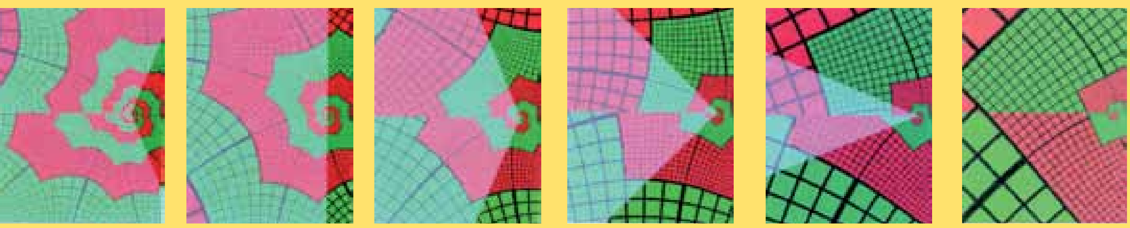
décroît de 1 à 0



*d)* α = 5//8 *e)* α = 4/8 *f)* α = 3/8 *g)* α = 2/8 *h)* α = 1/8 *i)* α = 0.0001

sée par Escher.

croît de 0 à 1



*d)* α = 3/8 *e)* α = 4/8 *f)* α = 5//8 *g)* α = 6/8 *h)* α = 7/8 *i)* α = 1

## La mise en équation des transformations

On représente un point $(x, y)$ du plan par le nombre complexe $z = x + iy$. Soit $r$ le module de $z$, et $\theta$ son argument. Pour envoyer le plan sur un cône, on utilise la transformation $z \mapsto z^{\alpha}$, que l'on peut voir en coordonnées polaires comme $(r, \theta) \mapsto (r^{\alpha}, \alpha\,\theta)$. Au départ, $\alpha = 1$. Ensuite, on fait décroître $\alpha$ vers 0. Mais on veut aussi envoyer le centre à l'infini. Donc, on va plutôt utiliser la formule :

$$z \mapsto Z = \frac{z^{\alpha}-1}{\alpha}.$$

On peut vérifier que cette formule envoie l'origine dans le point -1/alpha, et que le cercle de rayon 1 (donc de longueur $2\pi$) est envoyé sur un arc de cercle de longueur $2\pi$.

Quelle est la limite quand $\alpha = 0$? Précisément $z \mapsto Z = \log z$ (c'est une application de la règle de l'Hospital)! Au début, l'image d'Escher était invariante sous l'homothétie $z \mapsto zC$, pour $C = 1/256$. Que se passe-t-il pour l'image finale lorsqu'on a appliqué le logarithme? Comme $\log zC = \log z + \log C$, elle est devenue invariante sous une translation de période :

$$T_1 = \log C = -\log 256.$$

La deuxième période est $T_2 = 2\pi i$. Considérons le nombre complexe

$$A = \frac{T_2}{T_2 - T_1} = \frac{2\pi i}{2\pi i + \log 256}.$$

Multiplier $Z$ par $A$, c'est lui appliquer une homothétie dont le rapport est le module de $A$, suivie d'une rotation dont l'angle est l'argument de $A$. En appliquant la transformation inverse de la première transformation $z \mapsto Z = \log z$ (c'est-à-dire $Z \mapsto e^{z}$), on trouve $z' = e^{AZ}$. Faisons le calcul si $Z = \log z$ :

$$z' = e^{A \log z} = e^{\log z^{A}} = z^{A}.$$

L'image initiale était invariante sous l'homothétie $z \mapsto zC$, ce qui donne que la nouvelle image est invariante sous $z \mapsto C^{A}z$. Si on fait le calcul, le module de $C^{A}$ est environ 1/22,58 et l'argument 157,63 degrés, soit l'angle observé par Hendrik Lenstra.

# Artful Mathematics: The Heritage of M. C. Escher

## Celebrating Mathematics Awareness Month

*In recognition of the 2003 Mathematics Awareness Month theme "Mathematics and Art", this article brings together three different pieces about intersections between mathematics and the artwork of M. C. Escher. For more information about Mathematics Awareness Month, visit the website* `http://mathforum.org/mam/03/`*. The site contains materials for organizing local celebrations of Mathematics Awareness Month.*

## The Mathematical Structure of Escher's Print Gallery

*B. de Smit and H. W. Lenstra Jr.*

In 1956 the Dutch graphic artist Maurits Cornelis Escher (1898–1972) made an unusual lithograph with the title *Prentententoonstelling*. It shows a young man standing in an exhibition gallery, viewing a print of a Mediterranean seaport. As his eyes follow the quayside buildings shown on the print from left to right and then down, he discovers among them the very same gallery in which he is standing. A circular white patch in the middle of the lithograph contains Escher's monogram and signature.

What is the mathematics behind *Prentententoonstelling*? Is there a more satisfactory way of filling in the central white hole? We shall see that the lithograph can be viewed as drawn on a certain *elliptic curve* over the field of complex numbers and deduce that an idealized version of the picture repeats itself in the middle. More precisely, it contains a copy of itself, rotated clockwise by 157.6255960832... degrees and scaled down by a factor of 22.5836845286....

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**Figure 1. Escher's lithograph "Prentententoonstelling" (1956).**

### Escher's Method

The best explanation of how *Prentententoonstelling* was made is found in *The Magic Mirror of M. C. Escher* by Bruno Ernst [1], from which the following quotations and all illustrations in this section are taken. Escher started "from the idea that it must...be possible to make an annular bulge," "a cyclic expansion...without beginning or end." The realization of this idea caused him "some almighty headaches." At first, he "tried to put his idea into

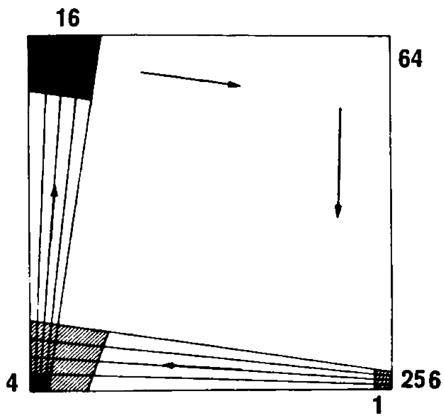


**Figure 2. A cyclic expansion expressed using straight lines.**

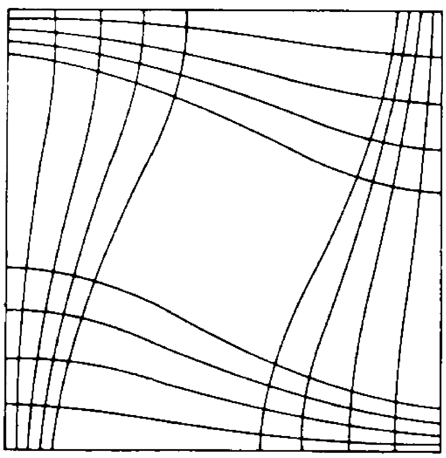

**Figure 3. A cyclic expansion expressed using curved lines.**

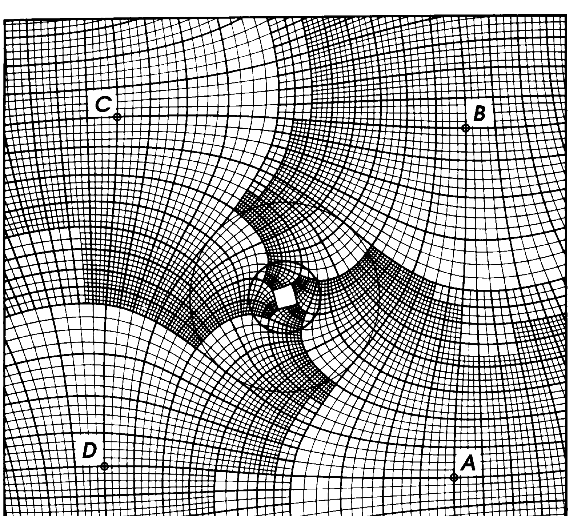


**Figure 4. Escher's grid.**

practice using straight lines [Figure 2], but then he intuitively adopted the curved lines shown in Figure [3]. In this way the original small squares could better retain their square appearance."

After a number of successive improvements Escher arrived at the grid shown in Figure 4. As one travels from $A$ to $D$, the squares making up the grid expand by a factor of 4 in each direction. As one goes clockwise around the center, the grid folds onto itself, but expanded by a factor of $4^4 = 256$.

The second ingredient Escher needed was a normal, undistorted drawing depicting the same scene: a gallery in which a print exhibition is held, one of the prints showing a seaport with quayside buildings, and one of the buildings being the original

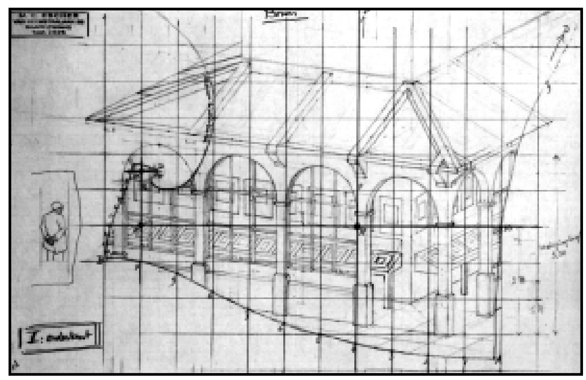

**Figure 5. One of Escher's studies.**

print gallery but reduced by a factor of 256. In order to do justice to the varying amount of detail that he needed, Escher actually made four studies instead of a single one (see [3]), one for each of the four corners of the lithograph. Figure 5 shows the study for the lower right corner. Each of these studies shows a portion of the previous one (modulo 4) but blown up by a factor of 4. Mathematically we may as well view Escher's four studies as a single drawing that is invariant under scaling by a factor of 256. Square by square, Escher then fitted the straight square grid of his four studies onto the curved grid, and in this way he obtained *Prentententoonstelling*. This is illustrated in Figure 6.

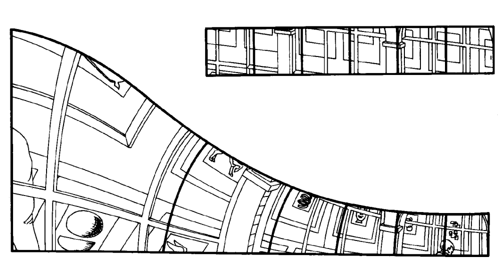

**Figure 6. Fitting the straight squares onto the curved grid.**

Below, we shall imagine the undistorted picture to be drawn on the complex plane $\mathbb{C}$, with 0 in the middle. We shall think of it as a function $f: \mathbb{C} \to \{\text{black, white}\}$ that assigns to each $z \in \mathbb{C}$ its color $f(z)$. The invariance condition then expresses itself as $f(256z) = f(z)$, for all $z \in \mathbb{C}$.

## A Complex Multiplicative Period

Escher's procedure gives a very precise way of going back and forth between the straight world and the curved world. Let us make a number of walks on his curved grid and keep track of the corresponding walks in the straight world. First, consider the path that follows the grid lines from $A$ to $B$ to $C$ to $D$ and back to $A$. In the curved world this is a closed loop. The corresponding path, shown in Figure 7, in the straight world takes three left turns, each time travelling four times as far as the previous time before making the next turn. It is not a closed loop; rather, if the origin is properly chosen, the end point is 256 times the starting point. The same happens, with the same choice of origin, whenever one transforms a single closed loop, counterclockwise around the center, from the curved world to the straight world. It reflects the invariance of the straight picture under a blow-up by a factor of 256.

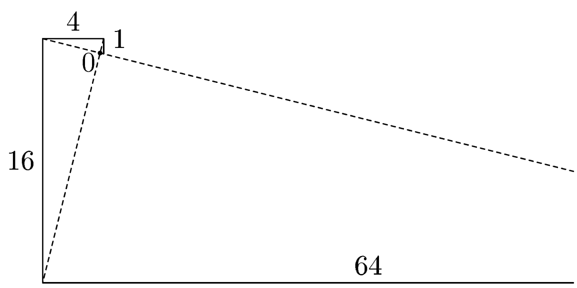


**Figure 7. The square $ABCD$ transformed to the straight world.**

No such phenomenon takes place if we do not walk around the center. For example, start again at $A$ and travel 5 units, heading up; turn left and travel 5 units; and do this two more times. This gives rise to a closed loop in the curved world depicted in Figure 8, and in the straight world it corresponds to walking along the edges of a $5 \times 5$ square, another closed loop. But now do the same thing with 7 units instead of 5: in the straight world we again get a closed loop, along the edges of a $7 \times 7$ square, but in the curved world the path does not end up at $A$ but at a vertex $A'$ of the small square in the middle. This is illustrated in Figure 9. Since $A$ and $A'$ evidently correspond to the same point in the straight world, any picture made by Escher's procedure should, ideally, receive the same color at $A$ and $A'$. We write *ideally*, since in Escher's actual lithograph $A'$ ends up in the circular area in the middle.

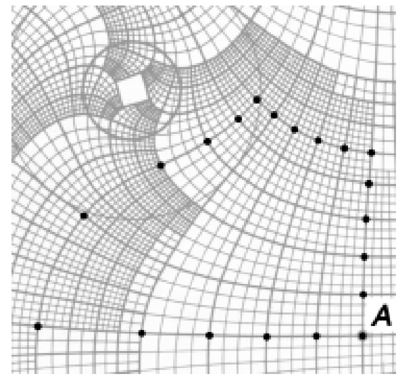


**Figure 8. A 5 x 5 square transformed to the curved world.**

We now identify the plane in which Escher's curved grid, or his lithograph, is drawn, also with $\mathbb{C}$, the origin being placed in the middle. Define $\gamma \in \mathbb{C}$ by $\gamma = A/A'$. A coarse measurement indicates that $|\gamma|$ is somewhat smaller than 20 and that the argument of $\gamma$ is almost 3.

Replacing $A$ in the procedure above by any point $P$ lying on one of the grid lines $AB$, $BC$, $CD$, $DA$, we find a corresponding point $P'$ lying on the boundary of the small square in the middle, and $P'$ will ideally receive the same color as $P$. Within the limits of accuracy, it appears that the quotient

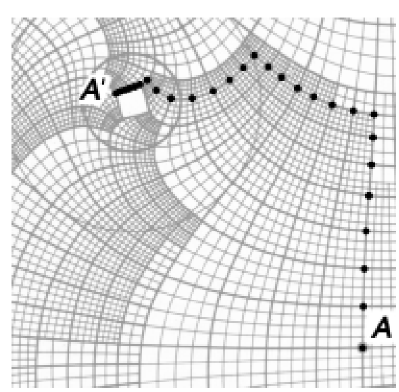


**Figure 9. A 7 x 7 square transformed to the curved world.**

**New Escher Museum**

In November 2002 a new museum devoted to Escher's works opened in The Hague, Netherlands. The museum, housed in the Palace Lange Voorhout, a royal palace built in 1764, contains a nearly complete collection of Escher's wood engravings, etchings, mezzotints, and lithographs. The initial exhibition includes major works such as *Day and Night*, *Ascending and Descending*, and *Belvedere*, as well as the *Metamorphoses* and self-portraits. The museum also includes a virtual reality tour that allows visitors to "ride through" the strange worlds created in Escher's works.

Further information may be found on the Web at `http://www.escherinhetpaleis.nl/`.

*Allyn Jackson*

$P/P'$ is independent of $P'$ and therefore also equal to $\gamma$. That is what we shall assume. Thus, when the "square" $ABCD$ is rotated clockwise over an angle of about 160° and shrunk by a factor of almost 20, it will coincide with the small central square.

Let the function $g$, defined on an appropriate subset of $\mathbb{C}$ and taking values in {black, white}, assign to each $w$ its color $g(w)$ in Escher's lithograph. If Escher had used his entire grid—which, towards the middle, he did not—then, as we just argued, one would necessarily have $g(P') = g(\gamma P')$ for all $P'$ as above, and therefore we would be able to extend his picture to all of $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ by requiring $g(w) = g(\gamma w)$ for *all* $w$. This would not just fill in the hole *inside* Escher's lithograph but also the immense area that finds itself *outside* its boundaries.

## Elliptic Curves

While the straight picture is periodic with a multiplicative period 256, an idealized version of the distorted picture is periodic with a complex period $\gamma$:

$$f(256z) = f(z), \qquad g(\gamma w) = g(w).$$

What is the connection between 256 and $\gamma$? Can one determine $\gamma$ other than by measuring it in Escher's grid?

We start by reformulating what we know. For convenience, we remove 0 from $\mathbb{C}$ and consider functions on $\mathbb{C}^*$ rather than on $\mathbb{C}$. This leaves a hole that, unlike Escher's, is too small to notice. Next, instead of considering the function $f$ with period 256, we may as well consider the induced function $\bar{f}: \mathbb{C}^*/\langle 256\rangle \to$ {black, white}, where $\langle 256\rangle$ denotes the subgroup of $\mathbb{C}^*$ generated by 256. Likewise, instead of $g$ we shall consider $\bar{g}: \mathbb{C}^*/\langle\gamma\rangle \to$ {black, white}. Escher's grid provides the dictionary for going back and forth between $f$ and $g$. A moment's reflection shows that all it does is provide a bijection $h: \mathbb{C}^*/\langle\gamma\rangle \xrightarrow{\sim} \mathbb{C}^*/\langle 256\rangle$ such that $g$ is deduced from $f$ by means of composition: $\bar{g} = \bar{f} \circ h$.

The key property of the map $h$ is elucidated by the quotation from Bruno Ernst, "…the original squares could better retain their square appearance": Escher wished the map $h$ to be a *conformal isomorphism*, in other words, an isomorphism of one-dimensional complex analytic varieties.

The structure of $\mathbb{C}^*/\langle\delta\rangle$, for any $\delta \in \mathbb{C}^*$ with $|\delta| \neq 1$, is easy to understand. The exponential map $\mathbb{C} \to \mathbb{C}^*$ induces a surjective conformal map $\mathbb{C} \to \mathbb{C}^*/\langle\delta\rangle$ that identifies $\mathbb{C}$ with the *universal covering space* of $\mathbb{C}^*/\langle\delta\rangle$ and whose kernel $L_\delta = \mathbb{Z}2\pi i + \mathbb{Z}\log\delta$ may be identified with the *fundamental group* of $\mathbb{C}^*/\langle\delta\rangle$. Also, we recognize $\mathbb{C}^*/\langle\delta\rangle$ as a thinly disguised version of the *elliptic curve* $\mathbb{C}/L_\delta$.

With this information we investigate what the map $h: \mathbb{C}^*/\langle\gamma\rangle \xrightarrow{\sim} \mathbb{C}^*/\langle 256\rangle$ can be. Choosing coordinates properly, we may assume that $h(1) = 1$. By algebraic topology, $h$ lifts to a unique conformal isomorphism $\mathbb{C} \to \mathbb{C}$ that maps 0 to 0 and induces an isomorphism $\mathbb{C}/L_\gamma \to \mathbb{C}/L_{256}$. A standard result on complex tori (see [2, Ch. VI, Theorem 4.1]) now implies that the map $\mathbb{C} \to \mathbb{C}$ is a multiplication by a certain scalar $\alpha \in \mathbb{C}$ that satisfies $\alpha L_\gamma = L_{256}$. Altogether we obtain an isomorphism between two short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_\gamma & \longrightarrow & \mathbb{C} & \stackrel{\exp}{\longrightarrow} & \mathbb{C}^*/\langle\gamma\rangle & \longrightarrow & 0 \\ & & \downarrow\alpha & & \downarrow\alpha & & \downarrow h & & \\ 0 & \longrightarrow & L_{256} & \longrightarrow & \mathbb{C} & \stackrel{\exp}{\longrightarrow} & \mathbb{C}^*/\langle 256\rangle & \longrightarrow & 0. \end{array}$$

Figure 10 illustrates the right commutative square.

In order to compute $\alpha$, we use that the multiplication-by-$\alpha$ map $L_\gamma \to L_{256}$ may be thought of as a map between fundamental groups; indeed, it is nothing but the isomorphism between the fundamental groups of $\mathbb{C}^*/\langle\gamma\rangle$ and $\mathbb{C}^*/\langle 256\rangle$ induced

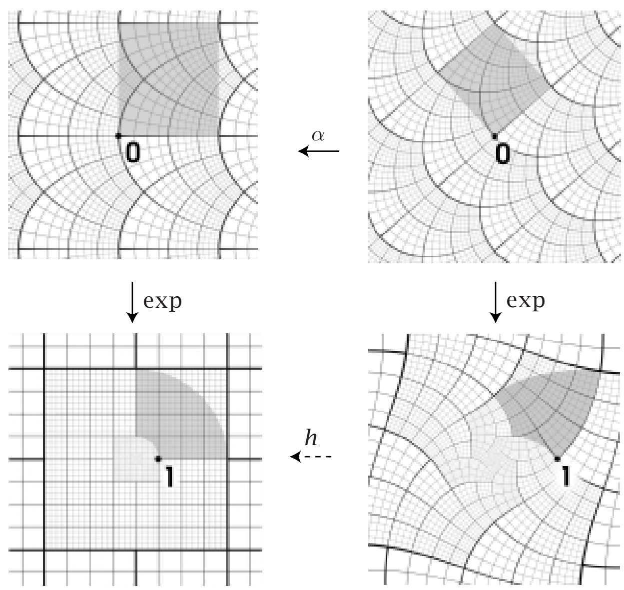


**Figure 10. The picture on the lower left, drawn on $\mathbb{C}^*$, is invariant under multiplication by $i$ and by 4. Pulled back to $\mathbb{C}$ by the exponential map it gives rise to a picture that is invariant under translation by the lattice $\frac{1}{4}L_{256} = \mathbb{Z}\pi i/2 + \mathbb{Z}\log 4$. Pulling this picture back by a scalar multiplication by $\alpha = (2\pi i + \log 256)/(2\pi i)$ changes the period lattice into $\frac{1}{4}L_\gamma = \mathbb{Z}\pi i/2 + \mathbb{Z}(\pi i \log 4)/(\pi i + 2\log 4)$. Since the latter lattice contains $2\pi i$, the picture can now be pushed forward by the exponential map. This produces the picture on the lower right, which is invariant under multiplication by all fourth roots of $\gamma$.**
**The bottom horizontal arrow represents the multivalued map $w \mapsto h(w) = w^\alpha = \exp(\alpha \log w)$; it is only modulo the scaling symmetry that it is well-defined.**

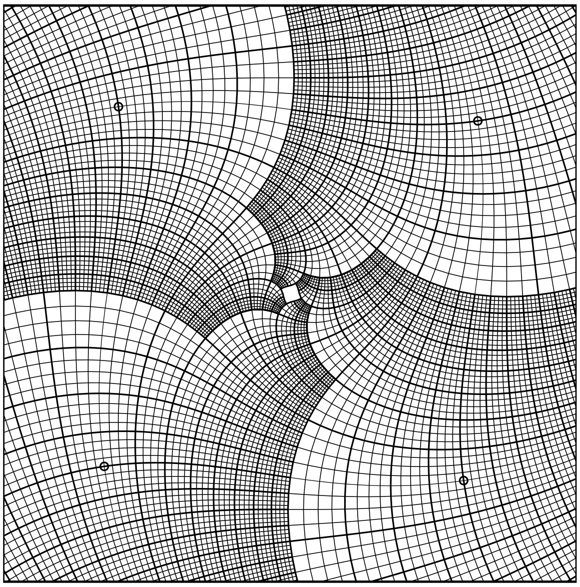

**Figure 11. The perfectly conformal grid.**

by $h$. The element $2\pi i$ in the fundamental group $L_\gamma$ of $\mathbb{C}^*/\langle\gamma\rangle$ corresponds to a single counterclockwise loop around the origin in $\mathbb{C}^*$. Up to homotopy, it is the same as the path $ABCDA$ along grid lines that we considered earlier. As we saw, Escher's procedure transforms it into a path in $\mathbb{C}^*$ that goes once around the origin and at the same time multiplies by 256; in $\mathbb{C}^*/\langle 256\rangle$, this path becomes a closed loop that represents the element $2\pi i + \log 256$ of $L_{256}$. Thus, our isomorphism $L_\gamma \to L_{256}$ maps $2\pi i$ to $2\pi i + \log 256$, and therefore $\alpha = (2\pi i + \log 256)/(2\pi i)$. The lattice $L_\gamma$ is now given by $L_\gamma = \alpha^{-1} L_{256}$, and from $|\gamma| > 1$ we deduce

$$\gamma = \exp(2\pi i(\log 256)/(2\pi i + \log 256))$$
$$\doteq \exp(3.1172277221 + 2.7510856371i).$$

The map $h$ is given by the easy formula $h(w) = w^\alpha = w^{(2\pi i + \log 256)/(2\pi i)}$.

The grid obtained from our formula is given in Figure 11. It is strikingly similar to Escher's grid. Our small central square is smaller than Escher's; this reflects the fact that our value $|\gamma| \doteq 22.58$ is larger than the one measured in Escher's grid. The reader may notice some other differences, all of which indicate that Escher did not perfectly achieve his stated purpose of drawing a conformal grid, but it is remarkable how close he got by his own headache-causing process.

## Filling in the Hole

In order to fill in the hole in *Prententententoonstelling*, we first reconstructed Escher's studies from the grid and the lithograph by reversing his own procedure. For this purpose we used software specially written by Joost Batenburg, a mathematics student at Leiden. As one can see in Figure 12, the blank spot in the middle gave rise to an empty spiral in the reconstructed studies, and there were other imperfections as well. Next, the Dutch artists Hans Richter and Jacqueline Hofstra completed and adjusted the pictures obtained; see Figure 13.

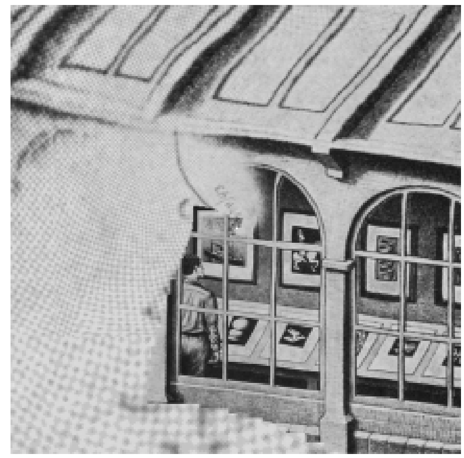

**Figure 12. Escher's lithograph rectified, by means of his own grid.**

When it came to adding the necessary grayscale, we ran into problems of discontinuous resolution and changing line widths. We decided that the natural way of overcoming these problems was by requiring the pixel density on our elliptic curve to be uniform in the Haar measure. In practical terms, we pulled the sketches back by the exponential function, obtaining a doubly periodic picture on $\mathbb{C}$; it was in that picture that the grayscale was added, by Jacqueline Hofstra, which resulted in the doubly periodic picture in Figure 14. The completed version of Escher's lithograph, shown in Figure 15, was then easy to produce.

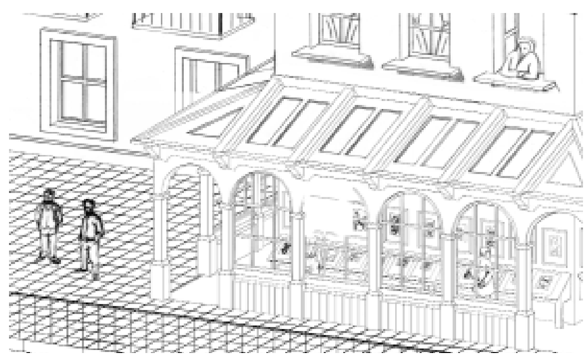

**Figure 13. A detail of the drawing made by Hans Richter and Jacqueline Hofstra.**

Other complex analytic maps $h: \mathbb{C}^*/\langle\delta\rangle \to \mathbb{C}^*/\langle 256\rangle$ for various $\delta$ give rise to interesting variants of *Prententententoonstelling.* To see these

**Figure 14. The straight drawing pulled back, by the complex exponential function, to a doubly periodic picture, with grayscale added. The horizontal period is log 256, the vertical period is $2\pi i$.**

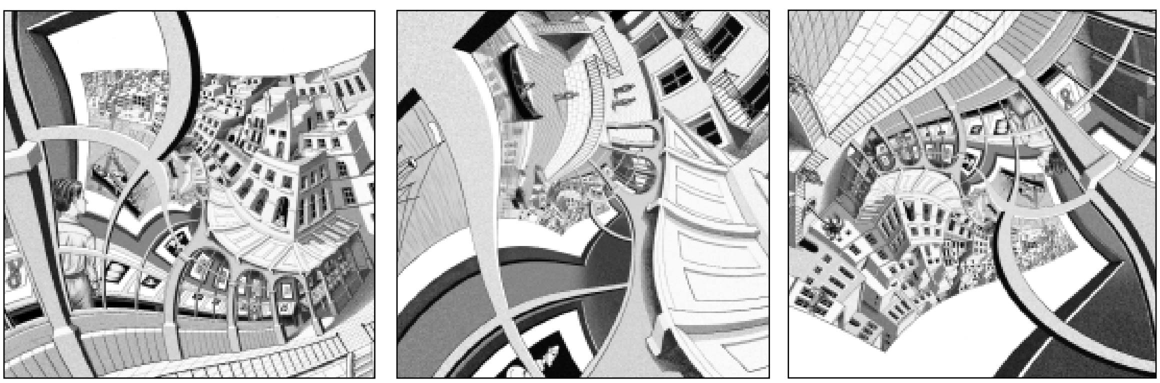

**Figure 15. The completed version of Escher's lithograph with magnifications of the center by factors of 4 and 16.**

and to view animations zooming in to the center of the pictures, the reader is encouraged to visit the website `escherdroste.math.leidenuniv.nl`.


## Acknowledgments

The assistance of Joost Batenburg, Cordon Art, Bruno Ernst, Richard Groenewegen, Jacqueline Hofstra, and Hans Richter is gratefully acknowledged. Cordon Art holds the copyright to all of M. C. Escher's works. The project was supported by a Spinoza grant awarded by the Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO).


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[1] Bruno Ernst, *De toverspiegel van M. C. Escher*, Meulenhoff, Amsterdam, 1976; English translation by John E. Brigham: *The Magic Mirror of M. C. Escher*, Ballantine Books, New York, 1976.

[2] J. Silverman, *The Arithmetic of Elliptic Curves*, Springer-Verlag, New York, 1986.

[3] E. Thé (design), *The Magic of M. C. Escher*, Harry N. Abrams, New York and London, 2000.

# Creating Repeating Hyperbolic Patterns—Old and New

*Douglas Dunham*

For more than a hundred years, mathematicians have drawn triangle tessellations in the Poincaré disk model of the hyperbolic plane, and some artists have found inspiration in these patterns. The Dutch graphic artist M. C. Escher (1898–1972) was most likely the first one to be so inspired. He used "classical" straightedge and compass constructions to create his hyperbolic patterns. More recently others have used computer graphics to create hyperbolic patterns. We will first explain how Escher drew the underlying tessellations for his patterns; then we will describe versions of the computer program that generated the design for the 2003 Mathematics Awareness Month poster.

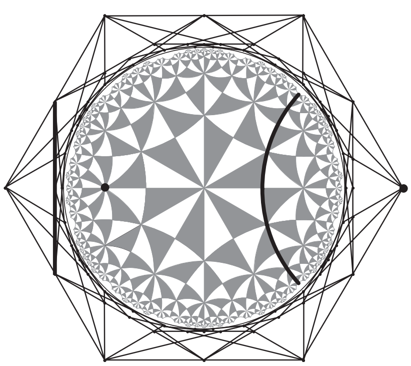

**Figure 1: The triangle tessellation of the Poincaré disk that inspired Escher, along with the "scaffolding" for creating it.**

## Escher's Methods

In 1958 H. S. M. Coxeter sent Escher a copy of his paper "Crystal symmetry and its generalizations" [Coxeter1]. In his reply Escher wrote, "…some of the text-illustrations and especially Figure 7, page 11, gave me quite a shock" [Coxeter2].

Escher was shocked because that figure showed him the long-desired solution to his problem of designing repeating patterns in which the motifs become ever smaller toward a circular limit. Coxeter's Figure 7 contained the pattern of curvilinear triangles shown in Figure 1 (but without the scaffolding). Of course, that pattern can be interpreted as a triangle tessellation in the Poincaré disk model of the hyperbolic plane, although Escher was probably more interested in its pattern-making implications.

The points of the Poincaré disk model are interior points of a bounding circle in the Euclidean plane. Hyperbolic lines are represented by diameters and circular arcs that are orthogonal to the bounding circle [Greenberg]. The Poincaré disk model is attractive to artists because it is conformal (angles have their Euclidean measure) and it is displayed in a bounded region of the Euclidean plane so that it can be viewed in its entirety. Escher was able to reconstruct the circular arcs in Coxeter's figure and then use them to create his first circle limit pattern, *Circle Limit I* (Figure 2a), which he included with his letter to Coxeter. Figure 2b shows a rough computer rendition of the *Circle Limit I* design. Escher's markings on the reprint that Coxeter sent to him show that the artist had found the centers of some of the circular arcs and drew lines through collinear centers [Schattschneider1].

The center of an orthogonal circular arc is external to the disk and is called its *pole*. The locus of all poles of arcs through a point in the disk is a line called the *polar* of that point [Goodman-Strauss]. The external dots in Figure 1 are the poles of the larger arcs, and the external line segments connecting them are parts of polars of the points of intersection of those arcs. Figure 1 shows one interior point and its polar as a large dot and a thick line on the left; a circular arc and its pole are similarly emphasized on the right.The external web of poles and polar segments is sometimes called the *scaffolding* for the tessellation. The fact that the polars are lines can be used to speed up the straightedge and compass construction of triangle tessellations. For example, given two points in the disk, the center of the orthogonal arc through them is the intersection of their polars.

Coxeter explained the basics of these techniques in his return letter to Escher [Roosevelt], although by that time Escher had figured out most of this, as evidenced by *Circle Limit I*. Like Escher, mathematicians have traditionally drawn triangle tessellations in the Poincaré disk model using straightedge and compass techniques, occasionally showing the scaffolding. This technique was something of a geometric "folk art" until the recent paper by Chaim Goodman-Strauss [Goodman-Strauss], in which the construction methods were finally written down.

For positive integers $p$ and $q$, with $1/p + 1/q < 1/2$, there exist tessellations of the hyperbolic plane by right triangles with acute angles $\pi/p$ and $\pi/q$. A regular $p$-sided polygon, or $p$-gon, can be formed from the $2p$ triangles about each $p$-fold rotation point in the tessellation. These $p$-gons form the regular tessellation $\{p, q\}$ by $p$-sided polygons, with $q$ of them meeting at each vertex. Figure 4 shows the tessellation $\{6, 4\}$ (with a central group of fish on top of it). As can be seen, Escher essentially used the $\{6, 4\}$ tessellation in *Circle Limit I.*

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Goodman-Strauss constructs the tessellation $\{p, q\}$ in two steps. The first step is to construct the central $p$-gon. To do this, he starts by constructing a regular Euclidean $p$-sided polygon $P$ with center $O$ that forms the outer edges of the scaffolding. Then he constructs the hyperbolic right triangle with a vertex angle of $\pi/p$ at $O$ and its hypotenuse along a radius of $P$ from $O$ to one of the vertices $A$ of $P$. The side of the right triangle through $O$ is part of a perpendicular bisector of an edge of $P$ containing $A$. The bounding circle is easily determined from the right triangle. The vertices of the entire central $p$-gon can then be constructed by successive (Euclidean) reflections across the radii and perpendicular bisectors of edges of $P$. The second step is to construct all the other $p$-gons of the tessellation. This *could* be done by first inverting all the vertices in the circular arcs that form the sides of the central $p$-gon, forming new $p$-gons, and then inverting vertices in the sides of the new $p$-gons iteratively as many times as desired. But Goodman-Strauss describes a more efficient alternative method using facts about the geometry of circles.

## Our First Method

In 1980 I decided to try to use computer graphics to recreate the designs in each of Escher's four *Circle Limit* prints (catalog numbers 429, 432, 434, and 436 in [Bool]). The main challenge seemed to be finding a replication algorithm that would draw each copy of a motif exactly once. There were two reasons for this. First, we were using pen-plotter technology then, so multiple redrawings of the same motif could tear through the paper. Efficiency was a second reason: the number of motifs increases exponentially from the center, and inefficient algorithms might produce an exponential number of duplications.

Moreover, I wanted the replication algorithm to build the pattern outward evenly in "layers" so that there would be no jagged edges. At that time my colleague Joe Gallian had some undergraduate research students who were working on finding Hamiltonian paths in the Cayley graphs of finite groups [Gallian]. I thought that their techniques could also be applied to the infinite symmetry groups of Escher's *Circle Limit* designs. This turned out to be the case, although we found the desired paths in two steps.

The first step involved finding a Hamiltonian path in the Cayley graph of the symmetry group of the tessellation $\{p, q\}$. This was done by David Witte, one of Gallian's research students. John Lindgren, a University of Minnesota Duluth student, implemented the computer algorithm, with me translating Witte's path into pseudo-FORTRAN [Dunham1].

The *Cayley graph* of a group $G$ with a set of generators $S$ is defined as follows: the vertices are just the elements of $G$, and there is an edge from $x$ to $y$ if $y = sx$ for some $s$ in $S$. Technically, this defines a directed graph, but in our constructions the inverse of every element of $s$ will also be in $S$, so for simplicity we may assume that our Cayley graphs are undirected. As an example, the symmetry group of the tessellation $\{p, q\}$ is denoted $[p, q]$. That symmetry group is the same as that of the tessellation by right triangles with angles $\pi/p$ and $\pi/q$. The standard set of generators for the group $[p, q]$ is $\{P, Q, R\}$, where $P$, $Q$, and $R$ are reflections across the triangle sides opposite the angles $\pi/p$, $\pi/q$, and $\pi/2$, respectively, in one such triangle.

There can be one-way or two-way Hamiltonian paths in the Cayley graphs of symmetry groups of hyperbolic patterns [Dunham3]. However, one-way paths are sufficient for our algorithms, so in this article "Hamiltonian path" will always denote a one-way Hamiltonian path.

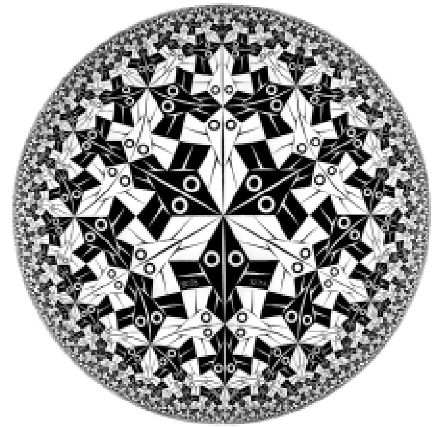

**Figure 2a: The original Escher print *Circle Limit I*.**



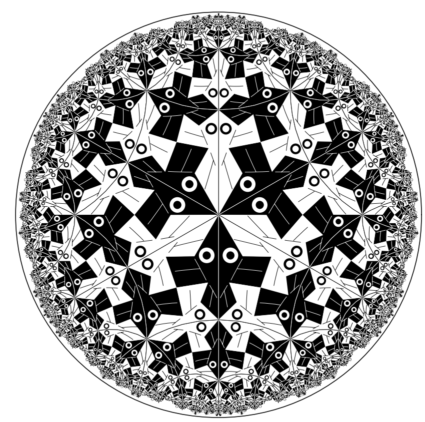

**Figure 2b: A computer rendition of the design in Escher's print *Circle Limit I*.**

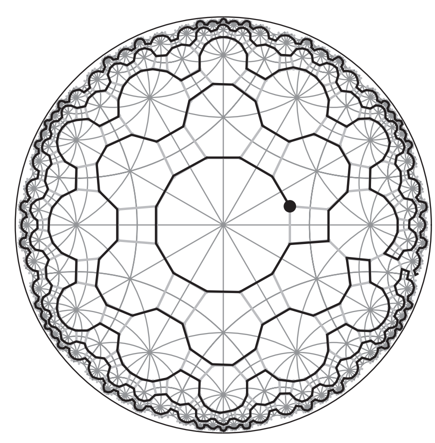

**Figure 3: The Cayley graph of the group [6,4] with a Hamiltonian path.**

There is a useful visual representation for the Cayley graphs of the groups $[p, q]$ and thus for their Hamiltonian paths. A *fundamental region* for the tessellation $\{p, q\}$ is a triangle that when acted on by the symmetry group $[p, q]$ has that tessellation as its orbit. This fundamental region can be taken to be a right

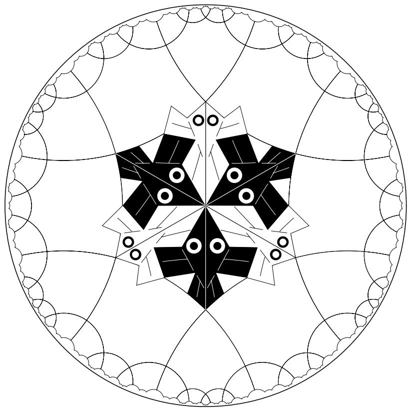

**Figure 4: The central "supermotif" for *Circle Limit I*.**

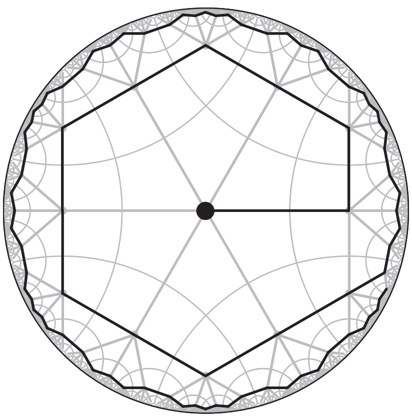

**Figure 5: A Hamiltonian path in a coset graph of [6,4].**

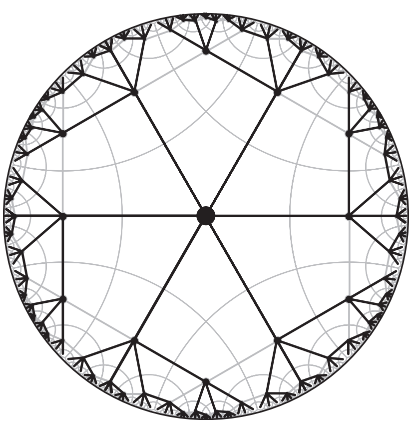

**Figure 6: A spanning tree in a coset graph of [6,4].**

triangle lying on the horizontal diameter to the right of center, with its $\pi/p$ vertex at the center of the disk. This triangle is labeled by the identity of $[p,q]$. Each triangle of the tessellation is then labeled by the group element that transforms the fundamental region to that triangle. Thus each triangle represents a group element. To represent an edge in the Cayley graph, we draw a line segment connecting the centers of any two triangles sharing a side. Thus, there are three line segments out of each triangle, each representing the reflection across one side.

Figure 3 shows a Hamiltonian path in the Cayley graph of the group $[6,4]$ with the standard set of generators. The heavy line segments, both black and gray, represent the Cayley graph; the light lines show the triangle tessellation. The Hamiltonian path consists of the heavy black line segments. Essentially the same path works for $[p,q]$ in general, though slight "detours" must be taken if $p = 3$ or $q = 3$ [Dunham3].

## Better Methods

A natural second step would have been to find Hamiltonian paths in the Cayley graphs of the symmetry groups of Escher's *Circle Limit* designs, which seemed possible since those symmetry groups were all subgroups of either $[6,4]$ or $[8,3]$. However, we took a slightly bigger step instead, which also works for any symmetry group that is a subgroup of $[p,q]$. This is the method used in the Mathematics Awareness Month poster. We first built up a *supermotif* (called the $p$-gon pattern in [Dunham1]) consisting of all the motifs adjacent to the center of the disk. The supermotif for the *Circle Limit I* design is shown in Figure 4, superimposed on top of the $\{6,4\}$ tessellation.

The second step then consisted of finding Hamiltonian paths in "Cayley coset graphs". Conceptually, we form the cosets of a symmetry group by the stabilizer $H$ of its supermotif. Then, by analogy with ordinary Cayley graphs, we define the vertices to be the cosets and say that there is an edge from $xH$ to $yH$ if $yH = sxH$ for some $s$ in $S$.

Again, there is a useful visual representation for these coset graphs. The vertices correspond to the $p$-gons in the tessellation $\{p,q\}$. The central $p$-gon is labeled by $H$, and any other $p$-gon is labeled by $xH$, where $x$ is any element of the symmetry group that maps the central $p$-gon to the other $p$-gon. In this case the generating set $S$ is composed of words in the generators of the symmetry group. For each side of the central $p$-gon, there is at least one word that maps the central $p$-gon across that side. Much as before, we represent edges of the graph as line segments between the centers of the $p$-gons. Figures 5 and 6 show the coset graph of a subgroup of $[6,4]$. Again, the graph edges are the heavy lines, either black or gray, and the light lines show the $\{6,4\}$ tessellation. The black graph edges in Figure 5 show a Hamiltonian path.

One shortcoming of this method is that the Hamiltonian path must be stored in computer memory. This is not a serious problem, since the path can be encoded by small integers. That method essentially works by forming ever-longer words in the generators from the edges of the Hamiltonian path. The transformations were represented by real matrices, which led to roundoff error after too many of them had been multiplied together to form the current transformation matrix. This was a more serious problem.

Both problems were cured by using recursion. What was required was to find a "Hamiltonian tree"—or more accurately, a spanning tree—in the coset graph. The tree is traversed on the fly by recursively transforming across certain sides of the current $p$-gon. Those sides are determined by fairly simple combinatorics [Dunham2] (there is an error in the recursive algorithm in [Dunham1]). With this method the path from the root (central $p$-gon) to the current $p$-gon is stored automatically in the recursion stack. Also, the recursion depth never gets deeper than a few dozen, so there is no noticeable roundoff error from multiplying transformations. The black graph edges in Figure 6 show a spanning tree in the coset graph of [6,4].

Other researchers have used different methods to generate a set of words in the generators of hyperbolic symmetry groups in order to replicate repeating hyperbolic designs. One method is to keep track of the transformations generated so far and iteratively add new transformations by multiplying all the previous transformations by the group generators, discarding duplicates. Then the desired pattern is generated by applying the final set of transformations to the motif. The techniques of automatic groups have been used to efficiently generate nonredundant sets of words. Silvio Levy has used this technique to create a computer rendition of Escher's *Circle Limit III* [Levy].

Like many mathematicians, I was immediately enthralled by M. C. Escher's intriguing designs when I first saw them more than thirty years ago. Several years later I started working in the field of computer graphics, and that medium seemed like an obvious one to use to produce Escher-like designs. When discussing the symmetry groups of Escher's hyperbolic patterns with Joe Gallian, it occurred to me that Hamiltonian paths could be used as a basis for an algorithm to generate such patterns. At this point everything had come together, and I could not resist the temptation to regenerate Escher's hyperbolic patterns with a graphics program. As described above, the students and I achieved this goal.

Having gone to the trouble of implementing a hyperbolic pattern program, I could not resist the further temptation of creating more hyperbolic patterns. I found inspiration in Escher's Euclidean repeating patterns. In constructing my patterns, I noticed I was paying attention to aesthetic issues such as shape and color. More of these hyperbolic designs can be seen in my chapter in the recent book *Escher's Legacy* [Schattschneider2]. When designing these patterns, I think of myself as working in the intersection of interesting mathematics, clever algorithms, and pleasing art.

## Acknowledgements

I would like to thank Doris Schattschneider for her considerable help, especially with the history of Escher and Coxeter's correspondence. I would also like to thank the many students who have worked on the programs over the years. Finally, I would like to thank Abhijit Parsekar for help with the figures.

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# Review of *M. C. Escher's Legacy: A Centennial Celebration*

*Reviewed by Reza Sarhangi*

M. C. Escher's Legacy: A Centennial Celebration
*Michele Emmer and Doris Schattschneider, Editors*
*Springer, 2003*
*450 pages, $99.00*
*ISBN 3-540-42458-X*

A few years ago an old friend who had visited a remote village in South America brought me a blanket as a souvenir. The purpose of this souvenir was to introduce me to a tessellation design of a foreign culture. To my amusement, the design was a work of Escher! The Dutch artist Maurits Cornelis Escher (1898–1972), who was inspired by the architecture of southern Italy and the pattern designs of North African Moors, is the source of inspiration and fascination of an incredible number of known and unknown artists in various cultures around the world.

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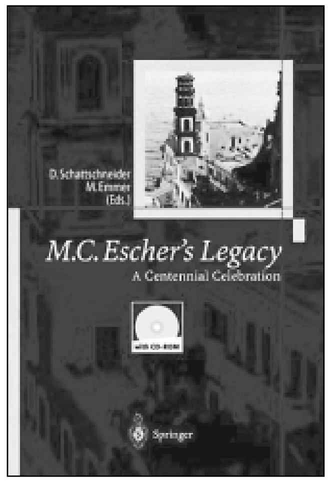
In the introduction of the book *M. C. Escher, The Graphic Work*, Escher describes himself: "The ideas that are basic to [my works] often bear witness to my amazement and wonder at the laws of nature which operate in the world around us. He who wonders discovers that this is in itself a wonder. By keenly confronting the enigmas that surround us, and by considering and analyzing the observations that I had made, I ended up in the domain of mathematics. Although I am absolutely without training or knowledge in the exact sciences, I often seem to have more in common with mathematicians than with my fellow artists." This description may very well explain why an international congress held in June of 1998 in Rome and Ravello, Italy, to celebrate the centennial of the birth of Escher was organized by two mathematicians, Michele Emmer and Doris Schattschneider. This conference resulted in an immensely interesting collection of articles presented in the book *M. C. Escher's Legacy, A Centennial Celebration*.




Emmer and Schattschneider, who also edited the book, are well known to mathematicians, scientists, and artists who seek aesthetic connections among disciplines. Schattschneider, a mathematics professor at Moravian College, has written numerous articles about tessellations and Escher's works as well as the book *Visions of Symmetry* (W. H. Freeman and Co., 1992), which describes Escher's struggles with the problem of dividing the plane, and his achievements. Emmer, a mathematics professor at the University of Rome, is one of the first in our time to call for a gathering of mathematicians and artists under one roof. He edited the book *The Visual Mind* (MIT Press, 1993) and created in the 1970s a series of videos about connections between mathematics and art. Several prominent scientists, such as geometer H. S. M. Coxeter and physicist and mathematician Roger Penrose, appeared in those videos. One of the videos, *The Fantastic World of M. C. Escher*, is still currently available.

This book is divided into three parts. The first part, "Escher's World", includes articles by authors with vast records of intellectual activities in connections among disciplines and in Escher's works. To mention a few, I should name Bruno Ernst, who wrote *The Magic Mirror of M. C. Escher* in 1976, and Douglas Hofstadter, who wrote *Gödel, Escher, Bach: An Eternal Golden Braid* in 1979. The book comes with a CD-ROM that complements many of the forty articles. The CD-ROM contains illustrations of artwork by contemporary artists who made contributions to the book, as well as several videos, a video-essay based on an Escher letter, some animations, and a puzzle.

Some articles in the first part of the book demonstrate dimensions of Escher that are mainly unknown to the public and therefore have not been extensively investigated. "Escher's Sense of Wonder" by Anne Hughes and "In Search of M. C. Escher's Metaphysical Unconscious" by Claude Lamontagne are two notable examples.

We also read about an Escher admirer and avid collector of his works, Cornelius Roosevelt, a grandson of American president Theodore Roosevelt. The article, by J. Taylor Hollist and Doris Schattschneider, illustrates a deep relationship between the artist and the art collector and describes how Roosevelt on many occasions played an advisory role for Escher in regard to the exhibitions and publications of the artist's works in the U.S. Cornelius Roosevelt donated his collection to the National Gallery of Art in Washington, DC. The collection is accessible to researchers, educators, and other interested individuals.

The second part, "Escher's Artistic Legacy", includes works of artists and authors such as S. Jan Abas, Victor Acevedo, Robert Fathauer, and Eva Knoll. There are also contributions by the artist-mathematician Helaman Ferguson and by Marjorie Rice, a homemaker without mathematical background who was inspired by the idea of pentagonal tessellations of the plane and who discovered four tilings unknown to the mathematics community at the time.

Not all the artists contributing to this part of the book were inspired by Escher; it is not a collection of papers written by the artist's disciples. Rather, the papers illustrate how the same sources for Escher's inspiration inspired these artists in a parallel way. Ferguson expressed that both he and Escher responded aesthetically to the same source: mathematics. Abas in a similar manner pointed to Islamic patterns as a common source. The reader not only becomes more familiar with the sources of inspiration but also has the chance to observe how the same sources resulted in different approaches and expressions in the final products of different artists. This helps the reader to reach a deeper understanding of Escher's works.

The third part, "Escher's Scientific and Educational Legacy", presents articles by the computer scientist Douglas Dunham, who is famous for performing tilings of the hyperbolic disk; the geometer H. S. M. Coxeter; the author and editor István Hargittai; Kevin Lee, who created the software utility Tessellation Exploration; and more.

Even though the majority of the articles in the book are accessible to a reader with little background in mathematics, there are works in part three that require familiarity with such concepts as geodesic, hyperbolic geometry; symmetry groups; and densest packing. These articles may be considered as a resource for research in undergraduate or graduate mathematics and mathematics education.

Escher is especially well known for two types of works: impossible structures (an idea borrowed from Penrose) and the regular division of the plane (influenced by Moorish artists of the Alhambra in Granada, Spain). Through impossible structures Escher challenges our concepts of the real world by tweaking our perception of dimensions. And through Escher's other trademark, the regular division of space and tessellations, he expresses ideas such as duality, symmetry, transformations, metamorphosis, and underlying relations among seemingly unrelated objects. The book presents several articles that address these two categories. However, some authors go further and study examples of Escher's works that go beyond these categories.

Bruno Ernst correctly suggests that the attraction of people to impossible structures and regular divisions of the plane has given a one-sided and incorrect image of Escher that ignores what he wanted to convey through his prints. Ernst notes that Escher was not a depicter of optical illusions, which are very different from the impossible constructions; this is a misconception on the part of the public. An example Ernst presents shows how the intention of the artist may be very different from the perception of the spectator.

Hofstadter writes about the first time he saw an Escher print. He was twenty years old in January 1966, and the print was in the office of Otto Frisch, who played a major role in unraveling the secrets of nuclear fission. Hofstadter was mesmerized by Escher's work *Day and Night*, showing two flocks of birds, one in white and one in black, flying in opposite directions. He asked Frisch, "What is this?" and Frisch replied, "It is a woodcut by a Dutch artist, and I call it 'Field Theory'. . . ."

The young man ponders the relationship to physics that this artwork may have sparked in Frisch's mind: "I knew that one of the key principles at the heart of field theory is the so-called CPT theorem, which says that the laws of relativistic quantum mechanics are invariant when three 'flips' are all made in concert: space is reflected in a mirror, time is reversed, and all particles are interchanged with their antiparticles. This beautiful and profound principle of physics seemed deeply in resonance with Frisch's Escher print . . . ."

The above example and more throughout the book illustrate the role of the artist Escher: a precise observer and a tireless and deep thinker whose prints not only bring us a sense of fascination and admiration but also provoke our intelligence to discover more relations beyond the scope and the knowledge of the artist through the meticulous and detailed presentations of symmetry, duality, paradox, harmony, and proportion. It is, then, not unrelated that Albert Falcon, a professor at the École des Beaux Arts in Paris, in a 1965 paper classified Escher among the "thinking artists", along with Da Vinci, Dürer, and Piero della Francesca.

I would like to end this review with the same poem that Emmer included at the end of one of his articles in this book. It is a Rubaiyat stanza by the Persian mathematician and poet of the eleventh century, Omar Khayyám.

Ah, moon of my delight, that knows no wane
The moon of Heaven is rising again,
How oft hereafter rising shall she look
Through this same garden after us in vain!

Although Escher himself is no longer among us, *M. C. Escher's Legacy*, like a garden of continually blooming flowers, allows us to appreciate his heritage anew.

### About the Cover

This month's cover exhibits the recently constructed extension, described in the article by Lenstra and de Smit, of M. C. Escher's extraordinary lithograph *Print Gallery*.

—*Bill Casselman*
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